VŠB – Technical University of Ostrava
Faculty of Electrical Engineering and Computer Science

PRELIMINARY COURSE IN PHYSICS

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Ostrava 2007

Contents

Preface

This course is held in the first semester of bachelor's study at the Faculty of Electrical Engineering and Computer Science. It is designed to help students improve their understanding of the fundamental principles of physics. After the introduction of a scientific approach to understanding nature, the role of measurement and basic units, it is mainly focused on the topics of mechanics of particle-like bodies, statics of rigid bodies and simple mechanics of fluids. The significance of conservation laws in physics is pointed out. Most first-year students know the basic concepts introduced here, but they are not able to apply them correctly, or to solve related numerical problems.

The great difficulty is also the mathematics. Students usually only have imperfect understanding of basic vector operations and no knowledge at all of calculus. As it is impossible to gain real insight in university physics without this apparatus and it is also necessary when solving practical problems, basic mathematical apparatus – rules of the work with vectors, derivatives and integrals – are introduced here as well. Of course, this can by no means substitute for the proper theory building provided in the later courses of mathematics. We rather try to demonstrate some physical applications of this mathematical apparatus and to help a student to get an inuitive grip of the basic operations. The emphasis is placed on practical skills of solving quantitative problems.

At the beginning of each chapter, there are several opening questions, explained later within the text. Then the list of the basic knowledge and skills that a student should acquire studying the chapter is included. The most important parts of the text are emphasised with an exclamation mark. There are many figures and several tables illustrating the text. Any theory of reasonable length is always followed by one or more sample problems, which are chosen to demonstrate how the physical problems should be analysed and then solved. Simple substitution of numeric values into an equation without insight has no value. At the end of each chapter or section there are several problems to be solved by a student. Many problems concerning daily life are included here. I hope they will encourage students to find their own approach and not only to check the results in the key in the back of the book.

As I am not a native English speaker, I am very grateful to Lesley Law who read the whole text and made many language corrections and improvements.

I would also appreciate notification by readers of any errors which may have crept into the text or the calculations, though every effort has been made to eliminate them.

Chapter 1

Introduction - a Scientific Approach to Understanding Nature, Measurement and Units, Newton's Laws

?

- How can we characterise things around us?
- How can we compare their characteristics, even at a distance?
- How to make sense of natural phenomena?
- What holds us on the ground?
- What takes place in the sky above?
- Why do comets always come back?

After finishing this chapter you should be able to

- name the SI base units and know their definitions,
- convert units,
- know the SI prefixes used to form decimal multiples and submultiples of SI units,
- formulate the basic principles of theory-building in science,
- formulate Newton's laws of motion and Newton's law of gravity,
- distinguish between mass and weight,
- (some of you) solve simple problems concerning planetary motions using Kepler's laws and Newton's law of gravity.

1.1 Scientific Approach to Understanding Nature

Physics and science in general attempt to analyze the real world around us. They are thus based on measurements, observations and comparisons. **Any physical theory must be**

verified by the comparison of collected data to the data predicted by the theory. Theories which contradict the experimental data must be corrected or rejected.

If there are two or more theories, that agree with the experiments equally well, there is yet another criterion used to decide among them, the so-called Occam's Razor (after the medieval English friar and philosopher William of Ockham). It is also known as the rule of simplicity: one should always choose the simplest explanation of a phenomenon, the one that requires the fewest leaps of logic. It also implies that one should not make more assumptions than the minimum needed to build a theory explaining the observed data. It follows that theories which cannot in principle be crosschecked by experiment are redundant, this principle is thus helpful to cut away metaphysical concepts. It underlies all scientific modeling and theory building. On the other hand, it should never be a dominant criterion of correctness of a theory, the crucial criteria must always be logical consistency and empirical evidence. The law of simplicity by no means implies that the entire world can be explained simply and in a way understandable for anyone without much effort. As Einstein phrased: "Everything should be made as simple as possible, but not simpler."

A theory can be tested either qualitatively (e.g. will a body subjected to a force change its velocity?) or quantitatively (what is the relation between the force and its velocity rate?). To compare a theory with an experiment quantitatively, we need to establish physical quantities and their units.

1.2 Measurements, Quantities and Units

When we measure a physical quantity, we do so by comparison to some standard. These standards should be adopted by general consensus. Each quantity is assigned its own unit, given by the selected standard (e.g. ell, inch, foot, yard, metre etc. for length). The need of such units for length, area, volume, weight, and time came up in ancient times. For instance, a former standard of the unit of length used in Czech countries, so called Prague ell (59.14 cm), was established in the 13th century. One can still see its etalon on the wall of the New Town Hall on the Charles Square in Prague and old town-halls of some other places. They were fixed there so that everybody could take the right measure.

The value of a physical quantity is the quantitative expression of a particular physical quantity as the product of a number and a unit, the number being its numerical value. Thus, the numerical value of a particular physical quantity depends on the unit in which it is expressed. For example

$$2\,\text{ft.} = 24\,\text{in} = 0.6096\,\text{m}$$

With the expansion of international traffic, trade and cooperation it became awkward using region-specific units. On 22 June 1799 two platinum standards representing the metre and the kilogram were deposited in the Archives de la République in Paris, which was the first step in the development of the present International System of Units. There are some base quantities, forming the basis of the International System of Units (SI), see table 1.1. We define other quantities in terms of them. For ease of understanding and convenience, some SI-derived units have been given special names and symbols, as shown in table 1.2.

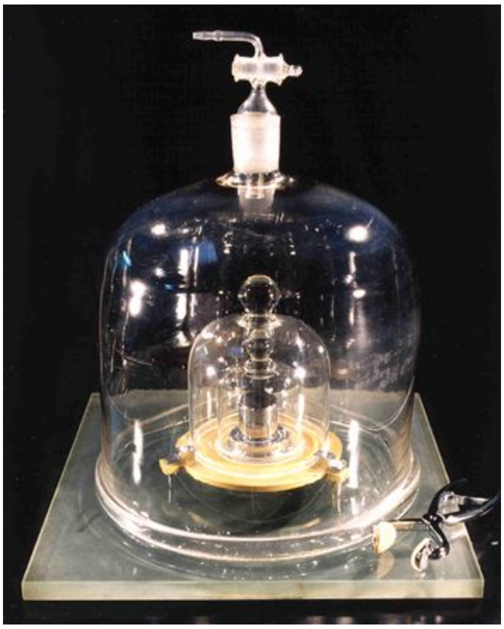


Figure 1.1: *Platinum-iridium prototype of kilogram* [10]

Some quantities have a wide range of values, compare for example masses of various objects:

- the Sun – $m \approx 2 \cdot 10^{30}$ kg
- Earth – $m \approx 6 \cdot 10^{24}$ kg
- Freedom of the Seas (the world's largest cruise ship) – $m = 1.6 \cdot 10^{8}$ kg
- a car – $m \approx 8 \cdot 10^{2}$ kg
- a human being – $m \approx 7 \cdot 10^{1}$ kg
- a ladybird – $m \approx 2 \cdot 10^{-5}$ kg
- a cell – $m \approx 10^{-21}$ kg
- a fullerene C_{60} – $m = 1.2 \cdot 10^{-24}$ kg
- a proton – $m = 1.7 \cdot 10^{-27}$ kg
- an electron – $m = 9.1 \cdot 10^{-31}$ kg.

It is therefore useful to use prefixes with the names of the units (see table 1.3). For example instead of $6 \cdot 10^{3}$ m or 6000 m (six thousand metres) one can say 6 km (6 kilometres).

! Note that kilogram is the only SI unit with a prefix as part of its name and symbol. Because multiple prefixes may not be used, in the case of mass the prefix names are used with the unit name gram instead of kilogram. For example 10^{-6} kg = 1 mg (one milligram), but not 10^{-6} kg = 1 μkg (one microkilogram).

There are some units outside the SI that are generally accepted for use with the SI, as units of

Quantity	Name	Symbol	Definition	Year
length	metre	m	The metre is the length of the path traveled by light in vacuum during a time interval of 1/299 792 458 of a second.	1983
mass	kilogram*	kg	The kilogram is the mass of the international prototype of the kilogram.	1889
time	second	s	The second is the duration of 9 192 631 770 periods of the radiation corresponding to the transition between the two hyperfine levels of the ground state of the cesium 133 atom.	1697
electric current	ampere	A	The ampere is that constant current which, if maintained in two straight parallel conductors of infinite length, of negligible circular cross-section, and placed 1 metre apart in vacuum, would produce between these conductors a force equal to $2 \cdot 10^{-7}$ newton per metre of length.	1946
thermodynamic temperature	kelvin	K	The kelvin is the fraction 1/273.16 of the thermodynamic temperature of the triple point of water.	1967
amount of substance	mole	mol	The mole is the amount of substance of a system which contains as many elementary entities as there are atoms in 0.012 kilogram of carbon-12.**	1971
luminous intensity	candela	cd	The candela is the luminous intensity, in a given direction, of a source that emits monochromatic radiation of frequency $540 \cdot 10^{12}$ hertz and has a radiant intensity in that direction of 1/683 watt per steradian.	1979

* The kilogram is the last unit still based on a prototype. It is awkward because it is not perfectly stable (its mass changes with time) and the values of the national copies cannot be monitored at the highest level of accuracy without being compared directly with it. However, alternative definitions of the kilogram are being discussed, e.g. a definitions based on Planck's constant or a fixed number of atoms.
** When the mole is used, the elementary entities must be specified and may be atoms, molecules, ions, electrons, other particles, or specified groups of such particles.

Table 1.1: *The International System of Units (SI)*

Derived quantity	Name	Symbol	Expression in terms of other SI units	Expression in terms of SI base units
plane angle	radian*	rad	$m \cdot m^{-1}$	1
solid angle	steradian*	sr	$m^2 \cdot m^{-2}$	1
frequency	hertz	Hz		s^{-1}
force	newton	N		$m \cdot kg \cdot s^{-2}$
pressure, stress	pascal	Pa	$N \cdot m^{-2}$	$m^{-1} \cdot kg \cdot s^{-2}$
energy, work, quantity of heat	joule	J	$N \cdot m$	$m^2 \cdot kg \cdot s^{-2}$
power, radiant flux	watt	W	$J \cdot s^{-1}$	$m^2 \cdot kg \cdot s^{-3}$
electric charge, quantity of electricity	coulomb	C		$s \cdot A$
electric potential difference,	volt	V	$W \cdot A^{-1}$	$m^2 \cdot kg \cdot s^{-3} \cdot A^{-1}$
capacitance	farad	F	$C \cdot V^{-1}$	$m^{-2} \cdot kg^{-1} \cdot s^4 \cdot A^2$
electric resistance	ohm	Ω	$V \cdot A^{-1}$	$m^2 \cdot kg \cdot s^{-3} \cdot A^{-2}$
electric conductance	siemens	S	$A \cdot V^{-1}$	$m^{-2} \cdot kg^{-1} \cdot s^3 \cdot A^2$
magnetic flux	weber	Wb	$V \cdot s$	$m^2 \cdot kg \cdot s^{-2} \cdot A^{-1}$
magnetic flux density	tesla	T	$Wb \cdot m^2$	$kg \cdot s^{-2} \cdot A^{-1}$
inductance	henry	H	$Wb \cdot A^{-1}$	$m^2 \cdot kg \cdot s^{-2} \cdot A^{-2}$
Celsius temperature	degree Celsius**	°C		K
luminous flux	lumen	lm	$cd \cdot sr$	cd
illuminance	lux	lx	$lm \cdot m^{-2}$	$m^{-2} \cdot cd$
activity (of a radionuclide)	becquerel	Bq		s^{-1}
absorbed dose	gray	Gy	$J \cdot kg^{-1}$	$m^2 \cdot s^{-2}$
dose equivalent	sievert	Sv	$J \cdot kg^{-1}$	$m^2 \cdot s^{-2}$
catalytic activity	katal	kat		$s^{-1} \cdot mol$

* The radian is an angle subtended at the centre of a circle by an arc equal in length to the radius of the circle. The steradian is the solid angle subtended at the centre of a sphere by an area on the surface of the sphere that is equal to the radius squared. The radian and steradian may be used to distinguish between quantities of a different nature but of the same dimension.
** It remains common practice to express a temperature rather as the difference of the thermodynamic temperature T from the reference temperature $T_0 = 273.15\,\mathrm{K}$, the ice point. This temperature difference is called a Celsius temperature, symbol t, and is defined by the quantity equation $t = T - T_0$. The degree Celsius is equal in magnitude to the kelvin.

Table 1.2: *SI derived units with special names and symbols*

Factor	Name	Symbol	Factor	Name	Symbol
10^{24}	yotta	Y	10^{-2}	centi	c
10^{21}	zetta	Z	10^{-3}	milli	m
10^{12}	tera	T	10^{-6}	micro	μ
10^{9}	giga	G	10^{-9}	nano	n
10^{6}	mega	M	10^{-12}	pico	p
10^{3}	kilo	k	10^{-15}	femto	f
10^{2}	hecto	h	10^{-18}	atto	a
10^{1}	deka	da	10^{-21}	zepto	z
			10^{-24}	yocto	y

Table 1.3: *SI prefixes*

- time - minute: $1\,\text{min} = 60\,\text{s}$, hour: $1\,\text{h} = 60\,\text{min} = 3600\,\text{s}$, and day: $1\,\text{d} = 24\,\text{h} = 86400\,\text{s}$;
- angle - degree: $1^\circ = \frac{\pi}{180}\,\text{rad}$, (angle) minute: $1' = \frac{1}{60}^\circ = \frac{\pi}{10800}\,\text{rad}$, and (angle) second: $1'' = \frac{1'}{60} = \frac{\pi}{648000}\,\text{rad}$;
- volume - litre: $1\,\text{L} = 1\,\text{dm}^3 = 10^{-3}\,\text{m}^3$

In everyday life in some countries there are still used some traditional units, especially of mass, length, volume, and temperature. See the following sample problem.

Sample problem 1.1 Australian Traditional Christmas Cake Recipe (from [13])
- 1/2 pound butter
- 1/4 pound white sugar
- 1/4 pound brown sugar
- 4 eggs
- 4 tablespoons brandy
- 1/2 pound raisins
- 1/2 pound sultanas
- 1/2 pound currants
- lemon peel and almonds to taste
- 10 ounces plain flour
- 1/2 teaspoon baking powder
- 1 teaspoon nutmeg
- 1 teaspoon cinnamon or allspice
- pinch of salt
- 1 tablespoon plum jelly

Directions:
Cream butter and sugar, add eggs. Sift in half of flour and half of fruit, mix, then add rest of ingredients. Bake in an 8" tin 3 1/2 to 4 hours at 300 degrees (Fahrenheit).
Task: Express all the quantities in SI units.
Hint: 1 pound = 454 grams, 1 ounce = 30 grams, 1 US fluid ounce = 30 millilitres = 2 US volume (liquid) tablespoons = 6 US volume teaspoons, 1" = 1 inch = 2.54 centimetres. To convert a Fahrenheit temperature t_F into Celsius t_C use the formula $t_\text{C} = \frac{5}{9} \cdot (t_\text{F} - 32)$.

Solution: We first convert 1/2 pound = 1/2 lb into a corresponding SI unit of mass, which is kilogram (kg).

$$1\,\mathrm{lb} = 454\,\mathrm{g} = 454 \cdot 10^{-3} \cdot (10^3\,\mathrm{g}) = 0.454\,\mathrm{kg},$$

so

$$\frac{1}{2}\,\mathrm{lb} = \frac{1}{2}\,\mathrm{lb} \cdot \frac{0.454\,\mathrm{kg}}{1\,\mathrm{lb}} = 0.227\,\mathrm{kg}.$$

Go on in the same way for the rest of the data.

More detailed information on SI, basic units, and their history can be found at [14].

Problem 1.1 The diameter of human hair is about 50 μm. Convert this value into metres.

Problem 1.2 The maximum speed limit on highways in the Czech Republic is 130 kilometres per hour. Convert it into metres per second.

Problem 1.3 On Easter Monday, 17. April 2006, the cost of a barrel of crude oil climbed up to $70,4 per barrel. What is the price per litre? (1 barrel = 42 US gallons, 1 US gallon = 3.785 liters).

Problem 1.4 A unit of power called horsepower has been defined by James Watt. It came from a driving wheel propelled by horses going round the circle. Watt estimated a horse pulled with a force of 180 pounds (gravitational force of 1 pound = 454 grams), at a speed of 181 feet (1 foot = 30.48 centimetres) per minute and rounded off the result to thousands of ft.-lbs./minute. Express 1 horsepower in watts.

1.3 Theory Building in Science via an Example of Uncovering Gravity

Once we have managed to describe certain aspects of the world around us using defined physical quantities and their units, we may attempt to discover natural patterns and then try to uncover basic laws governing natural processes and mutual interaction of objects. In fact, introduction of some new quantities and units may become effective during this process. New theory creation has some usual logical steps. Let's look at an example from the history of astronomy, one of the oldest branches of physics.

1. Phenomena invoking questions People were always interested in the daily movement of the Moon and the Sun, its annual movement, lunar phases, movement of the heavens with the stars during the night, and the periodic motion of the planets in the sky. They asked how it worked.

2. Observation (or experimentation, if possible), data collection and their organisation Astronomers and scientists collected empirical data, attempted to organise them, and tried to find some simple rules to explain them. There are several proofs of ancient interest in celestial mechanics. The Stonehenge complex (England) was being built during 3rd to 2nd millennium BC. It has long been studied for its possible connections with ancient astronomy. Although some theories have been found to be exaggerated, Stonehenge's builders must surely have had good knowledge of lunar and solar events. Ancient Egyptians also used astronomy in positioning the pyramids and in their calendar. It was based on lunar months with regard to the connection between the rise of Sirius and the annual flooding of the Nile.

3. Uncovering the rules The theories of the old-time astronomers were influenced by historical context and hampered by socioreligious prejudice, but conclusive facts given by the observed data finally gave birth to a relatively simple theory, which can be applied much more generally. From the ancient times let's mention just Plato, Aristotle, and Ptolemy (whose book Almagest was the standard text for astronomical study until Copernicus) and their more or less geocentric models.

Many centuries afterwards a Polish astronomer Nicolaus Copernicus (1473–1543) first proposed in his epochal book "De revolutionibus orbium coelestium" (On the Revolutions of the Celestial Spheres) that a rotating Earth revolving with the other planets around circular orbits about a stationary central Sun could account for the movement of the objects in the sky and it can do so in a simpler way than any geocentric model. Copernicus' theory was of extraordinary importance in the history of human knowledge, unfortunately, it was rejected by the church. This book was placed on the Index Librorum Prohibitorum in 1616 and only removed in 1835!

Figure 1.2: *Nicolaus Copernicus, portrait by R Cooper, and Galileo Galilei by Ottavio Leoni, 1624*

An Italian philosopher, Giordano Bruno (1548-1600) went even further. He suggested that space was boundless and that the sun and its planets were but one of a great number of similar systems, there even might be other inhabited worlds with rational beings like ourselves. He was very skeptical of the picture of the world and God as taught by the Church. For his predications Bruno was tried before the Inquisition, condemned and burned at the stake in 1600.

Another Italian, scientist Galileo Galilei (1564-1642) succeeded in constructing a telescope which magnified about thirty times. He turned it towards the heavens, and observed the moon was to possess hills and valleys and other features resembling those of our own globe. The planet Jupiter was found to have satellites, thus displaying a solar system in miniature, and supporting the doctrine of Copernicus. The spots on the sun, which Galileo perceived, served to prove its the rotation. In 1632 he published his "Dialogue Concerning the Two Chief World Systems", where he argued in favour of the Copernican system rather than the Ptolemaic system. One year later, Galileo was brought before the Inquisition, where he was, under threat of torture and death, forced to renounce all belief in Copernican theories, and was thereafter sentenced to house arrest for the rest of his life.

Figure 1.3: *Tycho de Brahe, portrait by Jacob de Gheyn, and Johannes Kepler, anonymous painting*

A Danish astronomer Tycho Brahe (1546 – 1601) collected extensive data based on exact observations of the motions of the planets, as they wander against the background of the stars (creditable work falling within the step 2 - observation, data collection and their organisation). He tried hard to create a theory of the Solar System. Nevertheless, he failed in his effort, because started from the misconception that all the planets except the Earth revolved about the Sun, the Sun itself going round the stationary Earth. All the same, his observations provided the groundwork for Kepler.

A German mathematician and astronomer Johannes Kepler (1571 – 1630) became Brahe's assistant. These two did not get along well, Brahe was afraid his young assistant could overtake him in the Solar System theory creation. So he set Kepler the task of understanding the puzzling trajectory of Mars. Brahe believed he would thus get enough time to work out his own theory. By an irony of fate it was mainly the Martian data that later allowed Kepler to formulate the correct laws of planetary motion. This data forced Kepler to realise that the orbits of the planets were not the exact circles assumed by his predecessors, but slightly "flattened circles" (ellipses).

Here are Kepler's laws, which constitute a milestone in the description of planetary motions.

I. All of the planets move in elliptical orbits, with the Sun as their focus. The eccentricity e of a planet's orbit measures how much it departs from a perfect circle (look at the picture 1.4). Orbits with eccentricity close to zero are almost circular, so are the planetary orbits. Comets and many asteroids follow more eccentric paths.

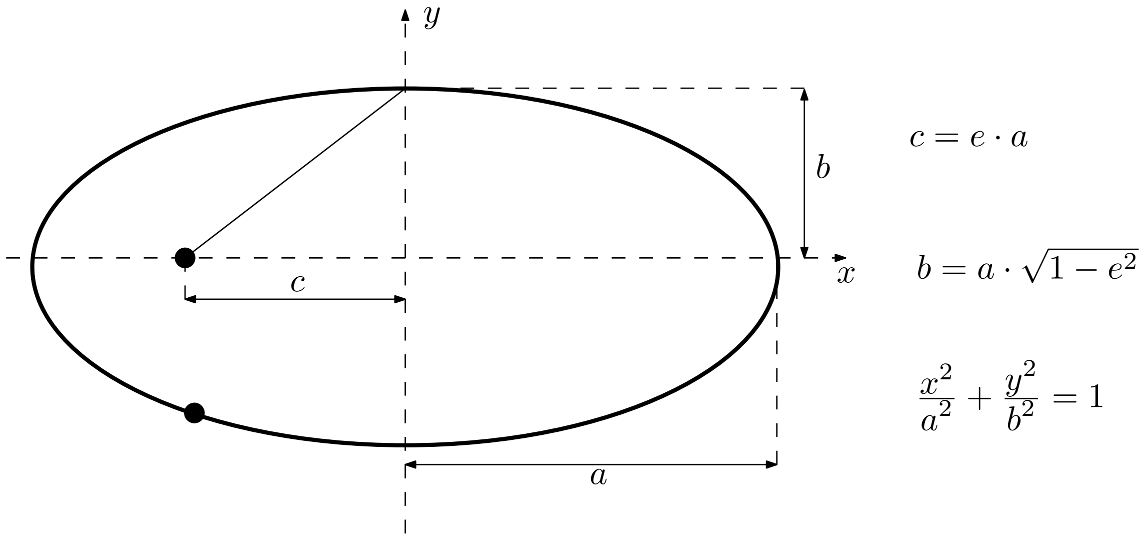


Figure 1.4: *An elliptical orbit of a planet*

II. The line joining the planet to the Sun sweeps out equal areas in equal times as the planet travels around the ellipse.

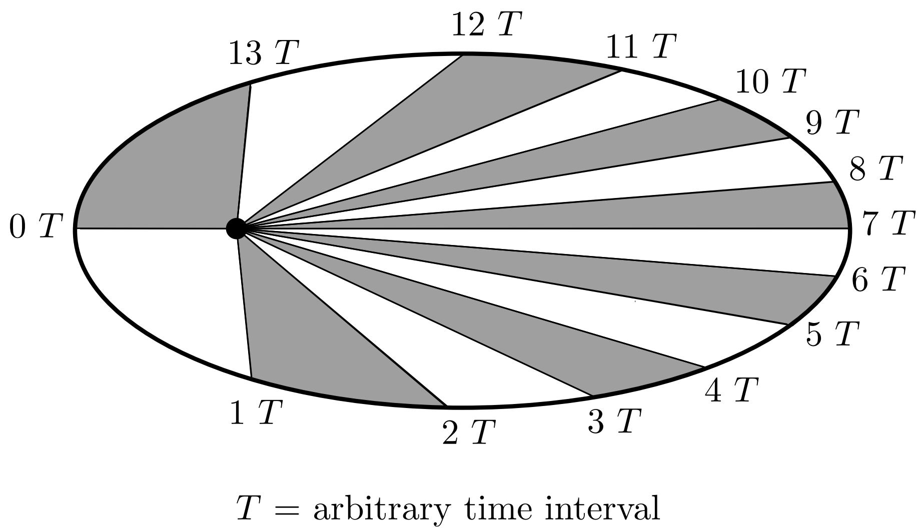


Figure 1.5: *The segments each have the same area and the time intervals are equal*

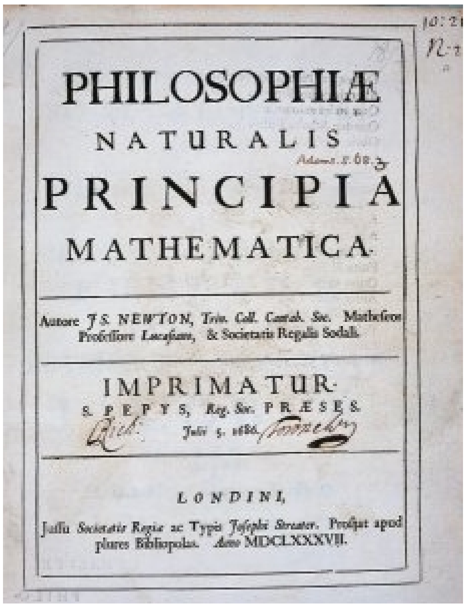
III. The ratio of the squares of the periods for two planets is equal to the ratio of the cubes of their semi-major axes.

$$\frac{T_1^2}{T_2^2} = \frac{a_1^3}{a_2^3}$$

In this equation T represents the period of revolution for a planet and a represents the length of its semi-major axis. The subscripts "1" and "2" distinguish quantities for planet 1 and 2 respectively.

Kepler's laws of planetary motions describe very well the motions of the planets around the Sun and their trajectories as observed from Earth. Nevertheless, they still do not **explain** why the planets obey such rules. To understand these rules fully, we need to find fundamental principles governing mutual interactions of objects and motion. Such laws would then be applicable not only to planetary motions, but to the motions of objects in general.

4. Discovering the fundamental principles This step was made by Sir Isaac Newton (1642–1727), an English mathematician and physicist, one of the greatest scientists in history of science. He was able to pick out the fruitful ideas of earlier scientists, supply the missing links and create a unified picture of how the universe works (corresponding to the level of knowledge of his day, of course). He formulated the three laws of motion, the most fundamental natural laws of classical mechanics, as well as the universal law of gravity (Philosophiae Naturalis Principia Mathematica - 1686-7). He concisely formulated how the interactions between the bodies, expressed by forces, influence their motion. The three laws of motion are included in Newton's first book of Principia.


PHILOSOPHIÆ
NATURALIS
PRINCIPIA
MATHEMATICA

Autore J S. NEWTON, Trin. Coll. Cantab. Soc. Matheseos Professore Lucasiano, & Societatis Regalis Sodali.

IMPRIMATUR.
S. PEPYS, Reg. Soc. PRÆSES.
Julii 5. 1686.

LONDINI,
Jussu Societatis Regiæ ac Typis Josephi Streater. Prostat apud plures Bibliopolas. Anno MDCLXXXVII.

Figure 1.6: *The title page of Isaac Newton's Principia, and his portrait by V. Rattier*

The second book deals with motion in resistant media, the third is devoted to gravitational force. Let us introduce these laws now, and we will explore them in much more detail later.

I. Newton's first law of motion:

! Every object in a state of uniform motion tends to remain in that state of motion unless an external force is applied to it.

$$\sum \boldsymbol{F} = \boldsymbol{0} \Rightarrow \boldsymbol{v} = \text{const.}$$

This means that if the resultant force (net force) acting on an object is zero, its velocity will naturally remain constant. Such an object will either move along a straight line at a constant speed, or it will remain motionless (maintain its zero speed).

The first law is based on Galileo's concept of inertia. An Italian mathematician, astronomer and philosopher Galileo Galilei (1564–1642) stated that an object in a state of motion possesses an "inertia" that causes it to remain in that state of motion unless an external force acts on it.

This statement is not so apparent taking into consideration our daily life on Earth. Rather we experience that some force is necessary to keep an object moving at constant velocity. If a moving body is not propelled, it will eventually stop and remain at rest. Just think of a car going on a perfectly straight road, or a puck sliding over the skating ring.

Unlike their predecessors, Galileo and later Newton revealed that a force is not required to keep a moving object in motion. On the contrary, it is a force which brings it to rest, a frictional force or a drag force. It took deep insight to reason that deceleration of moving objects is actually caused by the existence of frictional forces. This conclusion was based on the series of experiments with bodies moving on horizontal and inclined planes of various skid resistance.

The work of Galileo helped Newton to formulate also his second law.

II. Newton's second law of motion:

! If you place a force on an object, it will accelerate (change its velocity) in the direction of the force. This acceleration is directly proportional to the force and inversely proportional to the mass of the object.
The relationship between an object's mass m, its acceleration $\boldsymbol{a}$, and the applied force is $\boldsymbol{F}$ is

$$\boldsymbol{F} = m\,\boldsymbol{a}.$$

The acceleration of an object is the rate at which its velocity changes over time (we will come back to this quantity later in more detail). Note that acceleration and force are **vectors** (as indicated by their symbols being displayed in slant bold font).

– If an object moves along a straight line and we apply to it a force $\boldsymbol{F}$ in the direction of its motion, it speeds up. Its rate of speed will be

$$\frac{\Delta v}{\Delta t} = a = \frac{F}{m}.$$

If an applied force acts against its velocity, the object will equally slow down.

– If an object moves at uniform **speed**, not along a straight line but a curve, it undergoes acceleration as well, because its **velocity** vector changes. For instance a particle moving at a constant speed v along a circular path with the radius r possesses a centripetal acceleration. It is directed radially inward to the centre of the circle and its magnitude is

$$a = \frac{v^2}{r},$$

as we shall see later.

– Generally both the direction of a velocity vector and its magnitude (speed) can change.

We will pay attention to the subject of acceleration in chapter 4. There, we will study it in more detail and also discuss the formula for centripetal acceleration (I believe the gentle reader will tolerate this missing link for the moment).

III. Newton's third law of motion:

! For every action there is an equal and opposite reaction. The force exerted due to the mutual interaction upon the first object has the same size but the opposite direction to the related force acting on the second object.

$$\boldsymbol{F}_{12} = -\boldsymbol{F}_{21}$$

The third law was ingenious generalisation of experiments with solid bodies collisions, which had been performed by Wallis, Wren and Huygens. The third law states that in every interaction, there is a pair of forces acting on the two interacting objects. The action and the reaction always act simultaneously, but they do not cancel out, each of them acting on a different object.

Let's now recall a notorious hearsay about Newton and a falling apple and try to puzzle out, how could it inspire him? He realized an apple is attracted to the Earth and could speculate about how far from the Earth such an attractive force (gravity) operates. Maybe it is not just limited to the Earth and the objects on it. What if the same force extended to the Moon and beyond? It could then govern motion of the planets around the Sun as well. By such reasoning, Newton could come to the conclusion that any two objects in the Universe exert gravitational attraction on each other. This is an idea proven, together with the general law of gravity, in his third book of Principia.

This idea, of course, did not come to him as a bolt from the blue. The question of attractive forces between bodies was widely discussed among London scientists then. The assumption that attractive force is inversely proportional to the square of the distance was made by Newton's rival Robert Hook (1635-1703), who wrote it in a letter to Newton. Hook was not able to complete the law of gravity, but he later tried to take merit for its discovery. This ended in animosity between him and Newton, who then withdrew all the citations of Hook from his Principia (an example which certainly should not be followed) [12].

Using his laws of dynamics and II. and III. Kepler's laws of planetary motions Newton deduced a universal form of the law of gravitation. To illustrate the gist of the derivation, we show here just a simplified version. Suppose a planet with mass m follows a circular path around the Sun, whose mass is M. We denote r as the radius of the circle and v as the speed of the planet. Due to the III. Kepler's law the square of the period of a planet is proportional to the cube of its semi-major axes (radius in our simplified version).

$$T^2 \propto r^3.$$

The constant of proportionality is the same for all the planets in the Solar system. The period of our planet is $T = (2\pi r)/v$, which we can read as

$$\left(\frac{2\pi r}{v}\right)^2 \propto r^3.$$

A planet moving at a constant speed v along a circular path with the radius r undergoes a centripetal acceleration $a = v^2/r$. It follows

$$a \propto \frac{1}{r^2}.$$

Now consider the Newton's second law of motion. The acceleration of the planet must be induced by the force $F = ma$. In this case, it is a gravitational force. Therefore

$$F \propto \frac{m}{r^2}.$$

As gravity is universal, if the force exerted on the planet by the Sun is proportional to the mass of the planet m, the gravitational force exerted by the planet on the Sun must be proportional its mass M. Due to the Newton's third law both forces must have the same size. It implies the force exerted on the planet by the Sun is necessarily proportional to the mass of the Sun as well

$$F \propto \frac{mM}{r^2}.$$

To summarise, Newton's universal law of gravitation can be expressed as follows: Any two objects in the Universe exert gravitational attraction on each other, with the force having a universal form:

$$F = \kappa \frac{mM}{r^2}.$$

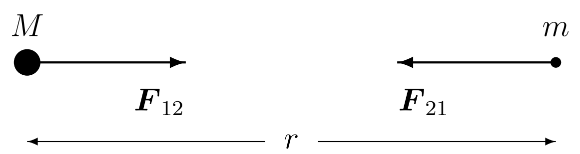


Figure 1.7: *Attractive gravitational forces between two mass objects*

The constant of proportionality κ is known as the universal gravitational constant, its value is $\kappa = 6.67 \cdot 10^{-11}\,\mathrm{N} \cdot \mathrm{m}^2 \cdot \mathrm{kg}^{-2}$. Here we meet the first of several universal constants in physics. "Universal constants" are thought to be the same at all places and all times, being intrinsic characteristics of the world around us.

5. Putting the theory under the test It is not very difficult to show that Newton's laws of motion and his universal law of gravitation imply Kepler's laws (including elliptical orbits of the planets), and the same laws determine motion of objects close to the Earth's surface as well. An early triumph of Newton's laws of motion and gravity was the prediction of Halley's comet's return in 1758, as well as its trajectory determination (see problem 1.8).

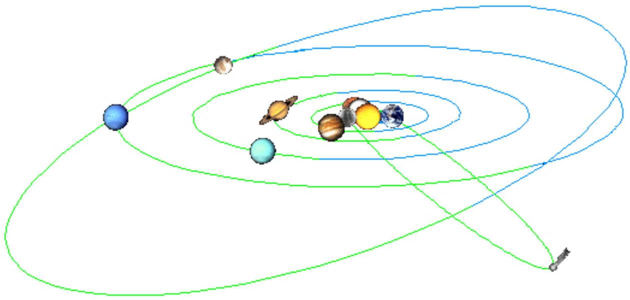


Figure 1.8: *Solar System, including Pluto and the orbit of Halley's comet*

Newton's laws have been successfully crosschecked by experiments and comparison with observations many times over and it has taken centuries and much technical progress till the limits of their applicability were discerned. It then gave rise to creation of new theories, but this is yet another story.

! Note: Close to the Earth's surface the gravitational force changes very slowly. The force exerted on bodies there can then be expressed in a more simple form

$$\boldsymbol{F_g} = m\,\boldsymbol{g},$$

where $\boldsymbol{g}$ is the vector of **free fall acceleration**

$$g \doteq \kappa \frac{M_\mathrm{Z}}{R_\mathrm{Z}^2} \doteq 9{,}8\,\mathrm{m}\cdot\mathrm{s}^{-2}.$$

Actually, $\boldsymbol{g}$ is a net acceleration, taking into account gravity and the rotation of Earth. It therefore slowly varies as you move along a terrestrial meridian. We will pay attention objects moving close to the Earth surface later.

Due to the existence of a gravitational field every body has some weight.

! The **weight** W of a body is equal to the magnitude of the gravitational force on the body,

$$W = F_g.$$

Be careful not to mistake an object's weight for its mass. The weight is related to mass by equation $W = mg$. Your weight on the Moon would be several times less than on Earth (see problem 1.12), but your mass would be the same.

The following problems are optional.

Problem 1.5 Look at the table 1.4 which shows the characteristics of selected orbits. Determine the object whose orbit departs the most significantly from a circle. What is the ratio of its major and semi-major axes?

Planetary Orbits			
Planet	Eccentricity	Perihelion (AU)	Aphelion(AU)
Mercury	0.206	0.31	0.47
Venus	0.007	0.718	0.728
Earth	0.017	0.98	1.02
Mars	0.093	1.38	1.67
Jupiter	0.048	4.95	5.45
Saturn	0.056	9.02	10.0
Uranus	0.047	18.3	20.1
Neptune	0.009	30.0	30.3
Pluto*	0.248	29.7	49.9
*According to a new official definition (2006), Pluto is no longer a planet, only a dwarf planet.			
Some other useful data: The astronomical unit, 1 AU $=1.496 \cdot 10^{11}$ m, the average distance from Earth to the Sun, Earth: $M_{\mathrm{E}} = 5.98 \cdot 10^{24}$ kg, $R_{\mathrm{E}} = 6378$ km, The Sun: $M_{\mathrm{S}} = 1.99 \cdot 10^{30}$ kg, $R_{\mathrm{S}} = 695000$ km, The Moon: $M_{\mathrm{M}} = 7.36 \cdot 10^{22}$ kg, $R_{\mathrm{M}} = 1740$ km, The distance from Earth to the Moon $a_{\mathrm{E-M}} = 3.82 \cdot 10^{8}$ m			

Table 1.4: *Selected Solar System characteristics*

Problem 1.6

Using the table 1.4 determine the ratio of the speed of the Earth in perihelion and aphelion. Then do the same for Mercury. Hint: Use Kepler's second law.

Problem 1.7 Look at the table 1.4 and determine periods of orbital motion of Mercury and Pluto. Hint: Use Kepler's third law and your knowledge about the Earth.

Problem 1.8 Halley's comet was first recorded by astronomers in 239 B.C and it comes close to the Sun regularly. Edmond Halley (1656–1742) successfully determined the comet's trajectory using Newton's laws and predicted its return in 1758, which was an early triumph of Newton's laws of motion and gravity. The comet's perihelion distance is 0.587 AU, its eccentricity 0.967. Last time it appeared close to the Sun was in 1986. When can we expect it again? Determine the lengths of its its major and semi-major axes.

Problem 1.9 What is the orbital speed of the Earth in perihelion?

Problem 1.10 Above the Earth's surface, there is a point at which you will encounter only half of its gravitational force as you would on the surface. How far above the surface is this point?

Problem 1.11 Suppose you go from the Earth straight to the Moon. How far from the centre of the Earth is there a point, where the gravitational force exerted on you by the Earth is the same as the gravitational force exerted on you by the Moon?

Problem 1.12 Compare the gravity on the Earth' surface and the surface of the Moon.

Chapter 2

Vectors and Some of Their Applications in Physics

?

How to express a quantity, whose direction, in addition to its size, is important?

How to add such quantities? What other operations on them can be useful?

How could Archimedes move a large ship with a single hand?

Swimming contest - who will be the winner?

What kind of force enables a car to go?

How to stand a ladder safely?

After finishing this chapter you should be able to

- explain the difference between scalar and vector quantities,
- add and subtract vectors graphically and mathematically
- find the scalar product and the vector product of two vectors, determine the magnitude of a vector and its direction
- express the gravitational force on a body of a certain mass and determine the reaction force of the base
- express the kinetic frictional force between two sliding surfaces and the maximum force of static friction
- add forces and determine equilibrium conditions,
- define the torque (momentum of force) including its unit,
- solve selected equilibrium problems.

2.1 Scalars and Vectors

Elementary physics deals with the two kinds of quantities

- **A scalar quantity** is determined by its value and the unit, e.g. mass, volume, time, pressure, energy, work and temperature.

$$m = 2\,\mathrm{kg}$$

- **A vector quantity** is determined not only by its value and the unit, but also by its direction, e.g. force, displacement, velocity, acceleration, momentum, angular momentum, torque etc.

$$\boldsymbol{F} = 2\boldsymbol{e}\,\mathrm{N},$$

  where $\boldsymbol{e}$ is a unit vector pointing in the direction of $\boldsymbol{e}$. Symbols of vectors are usually displayed in bold italics, e.g. $\boldsymbol{F}$. Alternatively, notation with an arrow, like $\vec{F}$ is used.

  Adding displacement vectors

Sample problem 2.1 A boat harboured at Porto Alegre (Sao Tome, an equatorial Atlantic island close to Africa) put forth upon the sea. It went 50 km southwards, then 30 km SSW, 50 km NW, and 30 km NNE. How far from home was it?

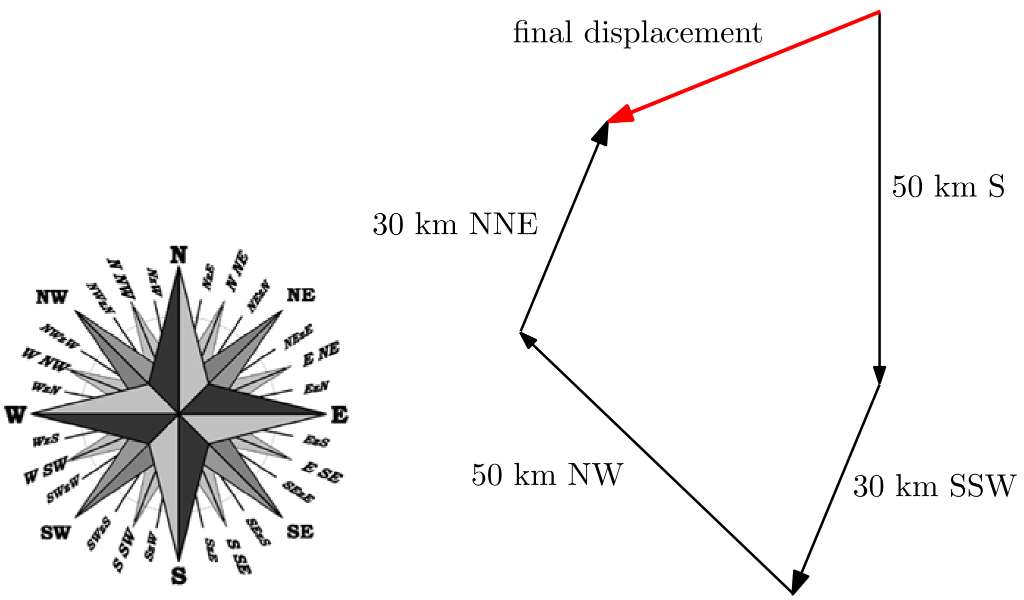


Figure 2.1: *A compass rose and the displacement vectors of the boat*

Solution: We can add all the displacement vectors geometrically as in figure 2.1. We draw the first vector with its proper length (using a convenient scale, of course, for instance 1:1 000 000) and direction. We then draw the second vector putting its tail at the head of the first vector, and so on. The final displacement is the vector with its tail at the tail of the first vector and its head at the head of the last. The distance from port equals its length, so we measure it. The result is: 38.3 km.

Some rules of vector calculations

- multiplication of a vector $\boldsymbol{A}$ by a scalar k

  $$a \cdot \boldsymbol{A}$$

  is a new vector. Its magnitude is the product of the magnitudes of $\boldsymbol{A}$ and a, its direction is the same as the direction of $\boldsymbol{A}$ if a is positive and the opposite if it is negative.

- the opposite vector of $\boldsymbol{A}$ is the vector

  $$-\boldsymbol{A} = (-1) \cdot \boldsymbol{A}$$

- commutative law (the order of addition does not matter)

  $$\boldsymbol{A} + \boldsymbol{B} = \boldsymbol{B} + \boldsymbol{A}$$

- associative law (if we add more than two vectors, they can be grouped in any order)

  $$(\boldsymbol{A} + \boldsymbol{B}) + \boldsymbol{C} = \boldsymbol{A} + (\boldsymbol{B} + \boldsymbol{C})$$

- vector subtraction (subtracting a vector is done by adding the vector opposite to it)

  $$\boldsymbol{A} - \boldsymbol{B} = \boldsymbol{A} + (-\boldsymbol{B})$$

Remark: Note that the second and the fourth vectors in our sample problem are opposite vectors. Their addition must therefore yield zero. Using commutative and associative laws we can see that the final displacement equals simply the addition of the first and the third vectors. Using the law of sines

$$\frac{a}{\sin\alpha} = \frac{b}{\sin\beta}$$

we can easily obtain the result by numerical computation.

2.2 A Rectangular Coordinate System

Adding vectors geometrically can be tedious and inaccurate. There must be a more hands-on way how to do it. It is possible to solve such problems using algebra. To do this, we must first introduce a coordinate system. We choose a right hand coordinate system (Cartesian coordinates). Imagine a rectangle room. We put the origin of our coordinate system into its lower left-hand corner. The x axis then goes along the intersection of the left wall and the floor, the y axis follows the intersection of the back wall and the floor and the z climbs up the intersection of these walls.

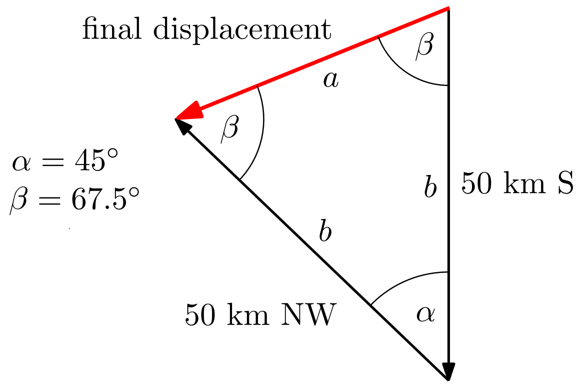


Figure 2.2: *A sketch of an exact triangle-based solution of the final displacement of the boat employing the rules of vector calculations*

We introduce three unit (with a magnitude of one) vectors, $\boldsymbol{i}$, $\boldsymbol{j}$, and $\boldsymbol{k}$, in the direction of axes x, y, and z respectively. These vectors are called Cartesian unit vectors (base vectors). Then we can write any vector $\boldsymbol{A}$ as

$$\boldsymbol{A} = A_x\boldsymbol{i} + A_y\boldsymbol{j} + A_z\boldsymbol{k},$$

the A_x, A_y, and A_z being the scalar components of the vector $\boldsymbol{A}$. They can be found as follows:

$$A_x = A\cos\alpha,\ A_y = A\cos\beta,\ A_z = A\cos\gamma,$$

where α, β, and γ are angles which the vector $\boldsymbol{A}$ makes with positive directions of the axes x, y, and z respectively. Once the base vectors are established, a simplified notation

$$\boldsymbol{A} = A_x\boldsymbol{i} + A_y\boldsymbol{j} + A_z\boldsymbol{k} = (A_x,\ A_y,\ A_z)$$

can also be used.

Rules of calculations using vector components

!

- Multiplying a vector by a scalar can be done by multiplying each of its components
$$a\boldsymbol{A} = (aA_x,\ aA_y,\ aA_z).$$
- Adding the vectors can be done by combining their components axis by axis
$$\boldsymbol{A} + \boldsymbol{B} = (A_x + B_x,\ A_y + B_y,\ A_z + B_z)$$
- Using the Pythagorean theorem twice we obtain the magnitude of the vector $\boldsymbol{A}$
$$A = \sqrt{A_x^2 + A_y^2 + A_z^2}.$$

Sample problem 2.2 Solve the task of the sailing boat (previous sample problem) using a coordinate system.

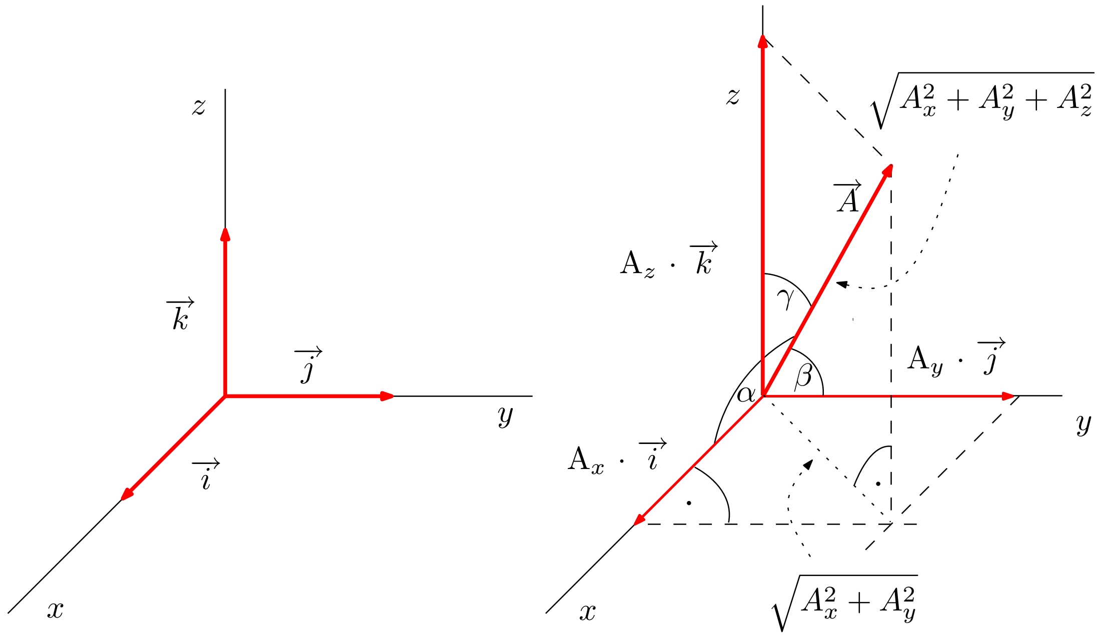


Figure 2.3: *Cartesian coordinate system, base vectors and vector components*

Solution: We choose a coordinate system, e.g. the origin in Porto Alegre, the x axis pointing westward and the y axes pointing southwards. The z axis of a right hand coordinate system would then point vertically upwards, but we do not need this axis at this time.

We express vector components of all the displacements

$$
\begin{aligned}
\boldsymbol{r_1} &= (0, 50)\,\text{km}, \\
\boldsymbol{r_2} &= \left(30 \sin\frac{\pi}{8}, 30 \cos\frac{\pi}{8}\right)\,\text{km}, \\
\boldsymbol{r_3} &= \left(50 \sin\frac{\pi}{4}, -50 \cos\frac{\pi}{4}\right)\,\text{km}, \\
\boldsymbol{r_4} &= \left(-30 \sin\frac{\pi}{8}, -30 \cos\frac{\pi}{8}\right)\,\text{km},
\end{aligned}
$$

and add the x and y components separately

$$\boldsymbol{r} = 50\left(\sin\frac{\pi}{4}, -1 + \cos\frac{\pi}{4}\right)\,\text{km}.$$

Then we find the magnitude of the net displacement

$$r = \sqrt{r_x^2 + r_y^2} = 38,3\,\text{km}.$$

2.3 Friction

As we saw in the first chapter, uncovering the existence of frictional forces played a key role in the formulation of Newton's first law. When two surfaces slide over each other,

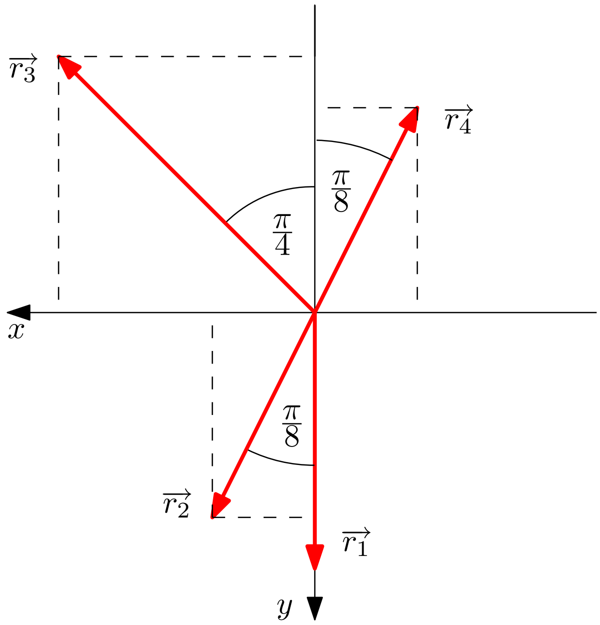


Figure 2.4: *The boat displacement vectors in the Cartesian coordinate system*

the friction is unavoidable (we need some counteracting force to keep objects in steady motion), but it is also very useful in our everyday life. If friction were totally absent, we could not walk or drive a car, nails and screws would spontaneously unfasten, and knots would untie.

Here we shall pay attention to the friction between two dry solid surfaces sliding over each other, or tending to slide, but there are more resistant forces in nature, such as friction in fluids or drag force. Frictional forces are contact forces that oppose relative motion of surfaces or parts of fluids. They are due to molecular adhesion and irregularities in the involved surfaces, in principle they originate from the electromagnetic forces and exchange force between atoms and molecules. The exact explanation based on interactions of these particles is complex. At present, there is no way to calculate friction accurately this way. Fortunately, there are some simple macroscopic characteristics of friction which provide a satisfactory base for the solutions of general practical problems.

Let's have a rectangular box - a cuboid. There are two different kinds of surface materials on it, each of them covering one of the pair of its opposite faces. We can also fix some additional weights in the middle of any face. Suppose we let the cuboid slide on various flat horizontal surfaces. Changing the contact face of the box, the surface on which it slides, and the normal force between the surfaces (adding weight to the top of the box), and measuring the external tension necessary to keep it in steady motion, e.g. using a spring balance, we can determine the value of the counteracting **kinetic frictional force**.

We can find the following rules.

1. The kinetic frictional force does **not** depend on the area of the touching surfaces.
2. The kinetic frictional force does **not** depend significantly on the relative speed of the surfaces.
3. The kinetic frictional force is proportional to the the magnitude of a normal force

$\boldsymbol{N}$ between the surfaces.

4. The proportionality constant between the friction and the normal force is due to the materials of the two surfaces in contact.

This proportionality constant μ_k is called the coefficient of kinetic friction. We can conclude:

! The magnitude of kinetic frictional force is the product of the coefficient of kinetic friction and the magnitude of a normal force between the surfaces,

$$F_\mathrm{fk} = \mu_\mathrm{k} N. \tag{2.1}$$

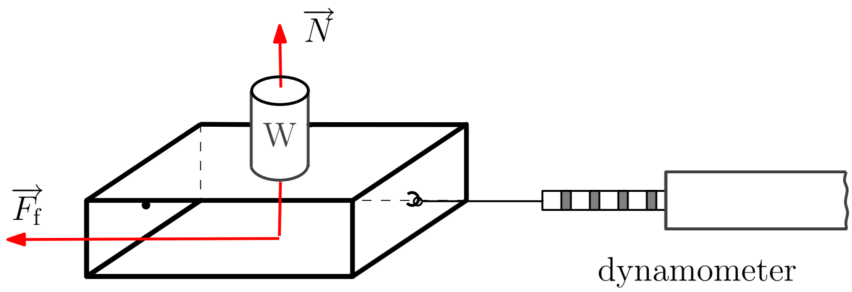


Figure 2.5: *Frictional force – we need to compensate for the kinetic frictional force to keep a cuboid in steady motion. To put a cuboid, originally at rest, in motion we have to surpass the static frictional force.*

Let's now modify our previous experiment and measure a horizontal force necessary to put a cuboid, originally at rest, in motion. We can find that a **static frictional force** is able to compensate for the counteracting horizontal tension force up to a certain value. This maximum value is again determined by the magnitude of a normal force $\boldsymbol{N}$ between the surfaces and a constant μ_s, called the coefficient of static friction, which depends on the materials in contact.

! The maximum magnitude of static frictional force is the product of the coefficient of static friction and the magnitude of a normal force between the surfaces,

$$F_\mathrm{fs} = \mu_\mathrm{s} N. \tag{2.2}$$

Note that equations 2.1 and 2.2 are **not** vector equations - the forces $\boldsymbol{F}_\mathrm{f}$ and $\boldsymbol{N}$ are perpendicular. The coefficients μ_k and μ_s are dimensionless and they must be determined experimentally. The results for various pairs of surfaces are included in tables such as table 2.1. As you can see, for most material combinations the static friction is higher than the kinetic friction. The friction can be reduced by applying lubricant to a surface, which is widely used in machinery.

In fact, more sophisticated methods of friction measurement are applied. Using a spring balance is a not very practical or exact method, especially when attempting to measure sliding (kinetic) friction. Even if we move the spring balance steadily to pull the box, we can observe slip-stick oscillations. It is due to the difference between the coefficients μ_k and μ_s. If the box is originally at rest, it remains at rest until the tension force in the spring gets large enough to overcome static friction. At this point, the box is released and jumps, the pulling force being greater than kinetic friction. Therefore the spring shortens and the pulling force diminishes under the value of kinetic friction, so the box sticks again, and so on. Such repetitive stick-and-slip can produce sounds, such as hte squeaking of hinges.

contact materials	μ_s	μ_k
steel on steel	0.7	0.6
lubricated steel on steel	0.1-02	0.05-0.1
glass on glass	0.9	0.40
ice on ice	0.1	0.03
tyre on dry road	0.9	0.7
tyre on wet road	0.6	0.4
tyre on snow	0.3	0.2
waxed ski on snow	0.05	0.1
human joints	0.1	0.03

Table 2.1: *Approximate values of coefficients of static and kinetic friction*

Sample problem 2.3 A boy pulls a sledge loaded with two of his friends across an ice-covered pond at constant velocity, the rope is horizontal. The total mass of the sledge and the burden is $m = 60\,\mathrm{kg}$. What is the magnitude of the tension force $\boldsymbol{T}$? The coefficient of kinetic friction for steel on ice is $\mu_k = 0.04$.

Solution: As the sledge moves at a constant velocity, the net force on the sledge must be zero (due to the Newton's first law). A plane of ice is horizontal. We sketch a picture of the sledge and all the forces exerted on it. They include

- the gravitational force $\boldsymbol{F}_g$,
- the normal force from the ice surface $\boldsymbol{N}$,
- the frictional force opposite to the direction of velocity $\boldsymbol{F}_{fk}$,
- the tension force due to the rope $\boldsymbol{T}$.

Therefore it must hold:

$$\boldsymbol{F}_g + \boldsymbol{N} + \boldsymbol{F}_{fk} + \boldsymbol{T} = \boldsymbol{0}$$

If we choose a coordinate system so that the x axis is parallel to the velocity and the y axis goes vertically upwards, we can rewrite this vector equation into equations for the components along these axes.

$$\begin{aligned} T - F_{fk} &= 0 \\ N - F_g &= 0. \end{aligned}$$

So $T = F_{fk}$. As we know that $F_{fk} = \mu_k N$ and it follows from the first equation that $N = F_g$ we have

$$T = \mu_k N = \mu_k F_g = \mu_k \cdot m \cdot g = 0.04 \cdot 60\,\mathrm{kg} \cdot 9.8\,\mathrm{m} \cdot \mathrm{s}^{-2} = 23.5\,\mathrm{N}$$

The boy pulls the sledge with the force $T = 23.5\,\text{N}$. Think over that this is only possible due to the fact that the coefficient of static friction between his shoes and ice is several times higher than the coefficient of kinetic friction between the sledge and ice.

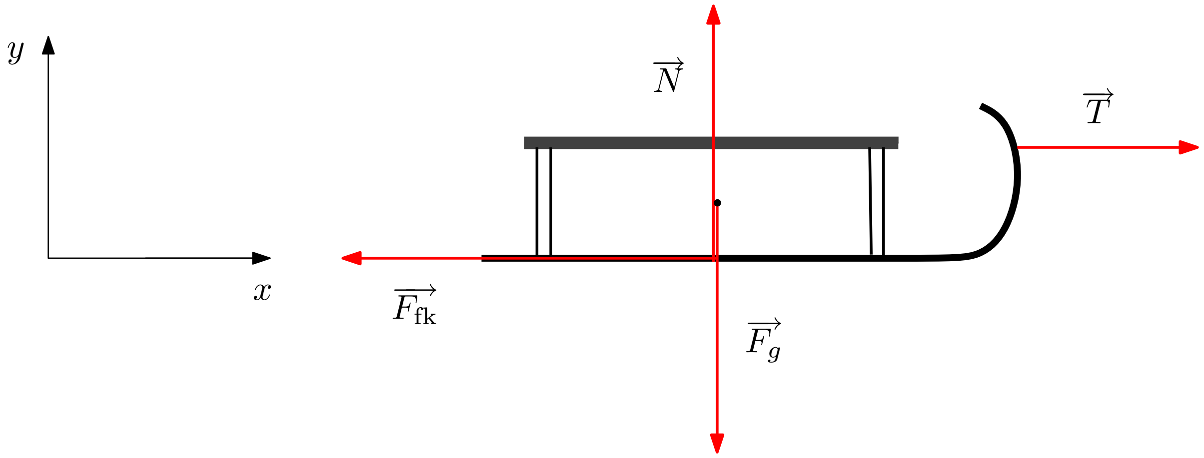


Figure 2.6: *Forces on a sledge sliding over the horizontal plane of an ice-covered pond*

Sample problem 2.4 Place a 1 Euro coin (its mass is exactly 7.5 g) on a closed hard-cover book lying on the table. Now start to open the cover very slowly until the coin starts sliding down. What is the coefficient of static friction between the coin and the book, if the coin starts sliding when the cover has been tilted at an angle $\alpha = 15°$? What was the frictional force on the coin at $\alpha' = 10°$?

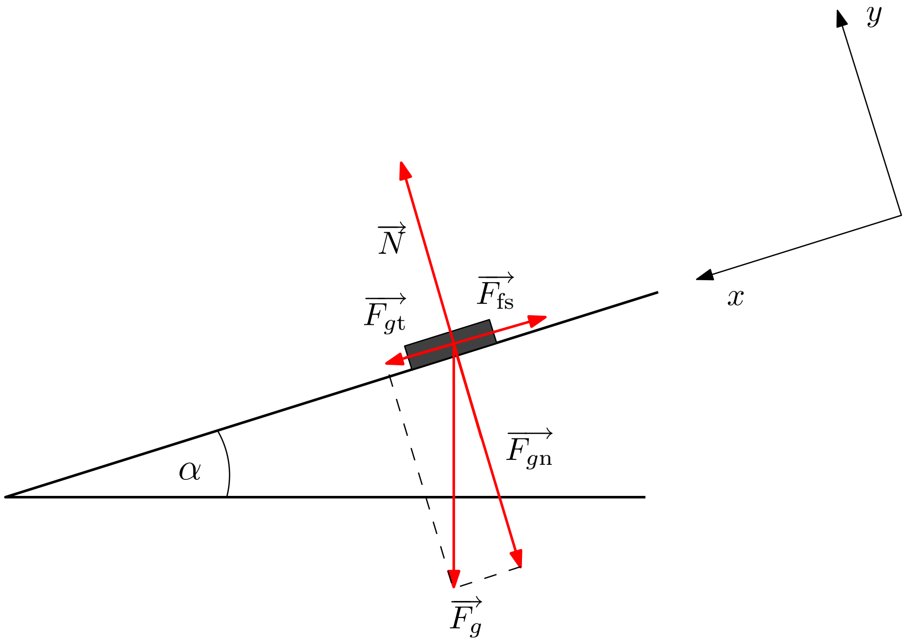


Figure 2.7: *Forces on a coin lying on an inclined plane*

Solution: We draw a picture of the coin on inclined plane and all the forces exerted on it. As in the previous sample problem, they include

- the gravitational force $\boldsymbol{F}_{\mathrm{g}}$,
- the normal force from the book surface $\boldsymbol{N}$,
- the frictional force opposite to the direction of intended motion $\boldsymbol{F}_{\mathrm{fs}}$.

Unlike the case of the sledge there is no additional tension force. While the coin is still at rest, it must hold:

$$\boldsymbol{F}_{\mathrm{g}} + \boldsymbol{N} + \boldsymbol{F}_{\mathrm{fs}} = \boldsymbol{0}$$

To solve the problem, we choose a coordinate system so that the x axis is parallel to the fall line of the inclined plane of the cover and the y axis is perpendicular to it. Both the tangential (x) and the normal (y) components of the gravitational force are non-zero:

$$\boldsymbol{F}_{\mathrm{g}} = (F_{\mathrm{gt}}, -F_{\mathrm{gn}}) = (F_{\mathrm{g}} \sin\alpha, -F_{\mathrm{g}} \cos\alpha),$$

the normal force from the cover is

$$\boldsymbol{N} = (0, N)$$

and the frictional force

$$\boldsymbol{F}_{\mathrm{fs}} = (-F_{\mathrm{fs}}, 0).$$

We can rewrite the vector equation into a couple of equations for the components along the axes:

$$\begin{aligned} N - F_{\mathrm{g}} \cos\alpha &= 0 \\ F_{\mathrm{g}} \sin\alpha - F_{\mathrm{fs}} &= 0. \end{aligned}$$

We also know that the maximum static friction is $F_{\mathrm{fs}} = \mu_{\mathrm{s}} N$. It follows from the first and the second equations respectively $F_{\mathrm{fs}} = \mu_{\mathrm{s}} N = \mu_{\mathrm{s}} F_{\mathrm{g}} \cos\alpha$ and $F_{\mathrm{fs}} = F_{\mathrm{g}} \sin\alpha$. Therefore

$$\mu_{\mathrm{s}} = \tan\alpha = \tan 15° \doteq 0.27$$

The coefficient of static friction is 0.27. To determine the frictional force at $\alpha' = 10°$, we must remember that a static frictional force adjusts itself to match the force that is trying to make the surfaces slide past each other. It therefore holds

$$F_{\mathrm{fs}} = F_{\mathrm{g}} \sin\alpha' = m \cdot g \sin\alpha' = 7.5 \cdot 10^{-3}\,\mathrm{kg} \cdot 9.8\,\mathrm{m} \cdot \sin 10° = 0.013\,\mathrm{N}.$$

Why does the following procedure:

$$\begin{aligned} F_{\mathrm{fs}} &= \mu_{\mathrm{s}} N = \mu_{\mathrm{s}} F_{\mathrm{g}} \cos\alpha' = \mu_{\mathrm{s}} \cdot m \cdot g \cos\alpha' = \\ &= 0.27 \cdot 7.5 \cdot 10^{-3}\,\mathrm{kg} \cdot 9.8\,\mathrm{m} \cdot \cos 10° = 0.020\,\mathrm{N} \end{aligned}$$

yield the wrong answer?

2.4 Pulley Systems

The outstanding ancient mathematician and physicist Archimedes (287–212 BC, Syracuse, Sicily) made a great contribution to practical utilisation of simple physical laws. One of his accomplishments was his creation of the lever and pulley systems. He invented many machines used in the defence of Syracuse when it was attacked by the Romans, such as catapults and claws to sink the enemy ships and curved mirrors, reflecting the sunlight to set ships on fire.

The story [15] says one day Archimedes boasted about the potential of the pulley systems he could design in face of the king Hieron, using the words: "Give me a place to stand on and I will move the Earth". The king challenged him to move a large ship in his arsenal, a ship that would take many men and great labour to move to the sea. On the appointed day, the ship was loaded with many passengers and a full cargo, and all watched to see if Archimedes could do what he said. He sat some distance away from the ship, pulled on the cord in his hand by degrees, and drew the ship along "as smoothly and evenly as if she had been in the sea." To find out how something like that was possible, we must first become acquainted with operating principles of pulley systems. Pulley assemblies are still used to move heavy objects at factories or on farms.

Sample problem 2.5 Suppose all the pulleys and ropes in the picture 2.8 can move without friction and their weights are negligible. All systems are at rest. Determine the external **tension force** exerted on each rope.

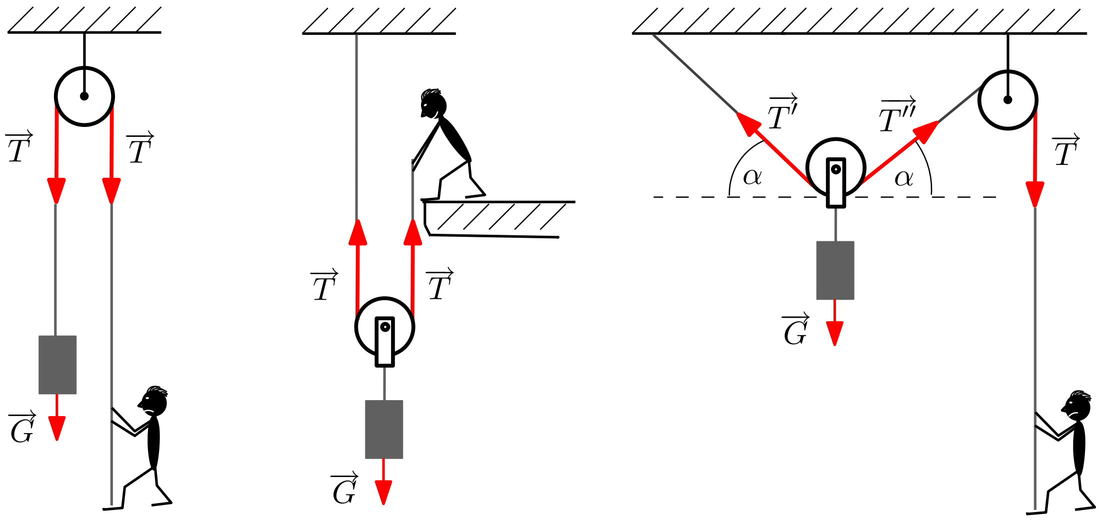


Figure 2.8: *Pulley systems*

Solution: As we disregard any friction in the pulley system, the tension force remains constant along the whole rope. It is then easy to draw the forces exerted on each pulley by all the segments of a rope and find their resulting force. The net force on each object, including gravitational force and/or the tension forces, must be (due to the Newton's first law) zero.

1. To keep the system in equilibrium the tension on the rope must be equal in magnitude to the weight of the burden $T = G$.

2. The tension vectors of the two parts of the rope exerted on the movable pulley add, so
$$T = \frac{G}{2}.$$

3. The forces exerted on a movable pulley are $\boldsymbol{T'}$, $\boldsymbol{T''}$, and $\boldsymbol{G}$, where $T' = T'' = T$. Therefore
$$T = \frac{G}{2\sin\alpha}.$$

Sample problem 2.6 Archimedes moving a ship may have used a pulley system such as in figure 2.9[16].

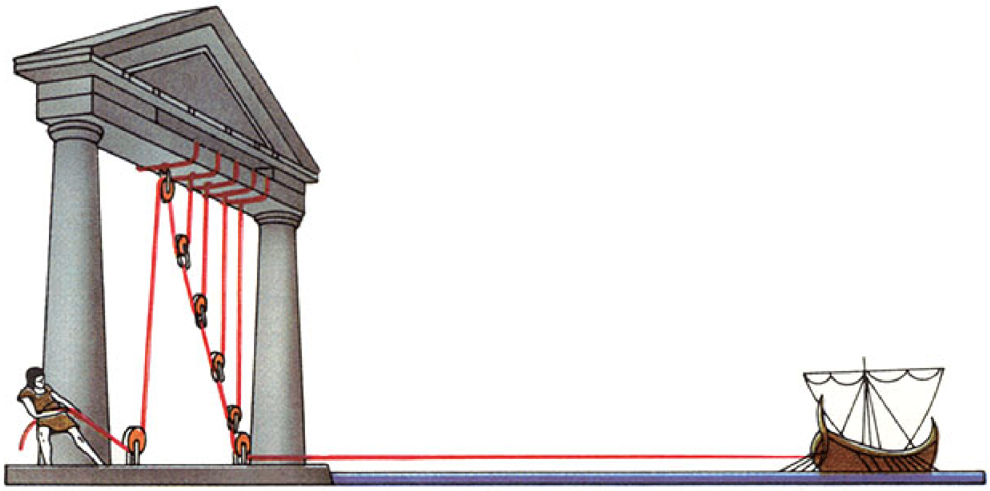


Figure 2.9: *A pulley system possibly designed by Archimedes to move a ship*

Assuming a ship was about $m = 80\,\mathrm{t}$ and a coefficient of static friction between the boat and the wooden dock was $\mu = 0.6$, how many movable pulleys would Archimedes have needed to move the ship? Suppose he could have exerted the maximum tensile force of $T = 500\,\mathrm{N}$.
Solution: The force applied on the ship must have been

$$F = \mu \cdot m \cdot g = 0.6 \cdot 80000 \cdot 9.8 \approx 470\,\mathrm{kN},$$

so the pulley system must have multiplied the the force of Archimedes almost 1000 times. Was that possible? It follows from the picture that (if we neglect the angle between the ropes, which can be minimised) each movable pulley halves the necessary tensile force invoked by Archimedes. Therefore he needed at least n movable pulleys, where:

$$\begin{aligned} T &\geq \frac{F}{2^n} \\ 2^n &\geq \frac{F}{T} \\ n\ln 2 &\geq \ln\frac{F}{T} = \ln F - \ln T \\ n &\geq \frac{\ln(\mu \cdot m \cdot g) - \ln T}{\ln 2} = \frac{\ln(0.6 \cdot 80000 \cdot 9.8) - \ln 500}{\ln 2} = 9.9 \end{aligned}$$

Archimedes needed just 10 movable pulleys to move a heavy ship.

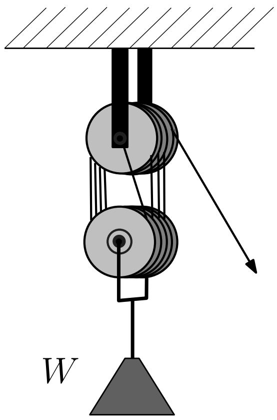


Figure 2.10: *A pulley block designed to lift heavy objects – each extra loop adds the double of the force by a person.*

Problem 2.1 If $\boldsymbol{A} = (20, 50, 30)$ and $\boldsymbol{B} = (-20, 10, 0)$, determine $\boldsymbol{A} + \boldsymbol{B}$.

Problem 2.2 Determine the magnitude of a vector $\boldsymbol{A} = (20, 30, 40)$

Problem 2.3 Determine the angles α, β and γ that the vector $\boldsymbol{A} = (20, 30, 40)$ makes with the positive directions of the x, y and z axes respectively.

Problem 2.4 Suppose all the pulleys and ropes in figure 2.11 can move without friction and their weights are negligible. All systems are at rest. Determine the external **tension force** exerted on each rope.

Problem 2.5 A pulley system (in figure 2.12) is used in a factory to move heavy workpieces. It is composed of one fixed pulley, one remotely controlled pulley horizontally movable and an electric drive. As the assembly moves, the total force on the fixed pulley from the tension in the cable changes. Determine its magnitude for a) $\alpha = 120°$ and b) $\alpha = 30°$. The weight of a workpiece is $W = 1\,\mathrm{kN}$.

Problem 2.6 A typical value of a drag force on a car going at a speed of $180\,\mathrm{km \cdot h^{-1}}$ is about $F_\mathrm{d} = 800\,\mathrm{N}$. The total mass of the car is $m = 800\,\mathrm{kg}$. What kind of force

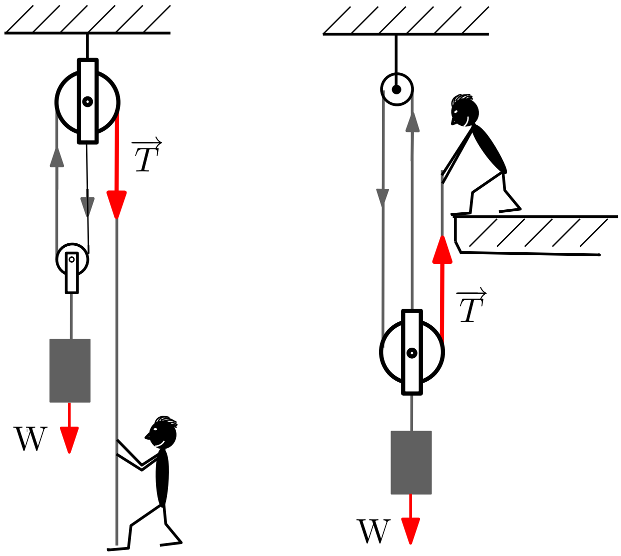


Figure 2.11: *Pulley systems from exercise 2.4*

compensates for the drag force and enables a car to maintain its speed? What is the minimum value of the coefficient of static friction between tyres and the road, which makes it possible?

Problem 2.7 A woman pulls a sledge loaded with her children across an ice-covered pond at constant velocity. The rope is inclined at an angle α from horizontal. The total mass of the sledge and the burden is m. What is the magnitude of the tension force $\boldsymbol{T}$? The coefficient of kinetic friction for steel on ice is μ_k.
Hint: Note that the normal force between the surfaces is not equal to the total weight of sledge.

Problem 2.8 In figure 2.13, a block of mass $M = 6\,\mathrm{kg}$ and an unknown weight are connected by a string of negligible mass over a massless and frictionless pulley. Determine the interval of possible mass m of weight if you know the system remains at rest. The plane is inclined by $\alpha = 30°$ from horizontal. The coefficient of static friction between the block and the inclined plane is $\mu_s = 0.5$.

Problem 2.9 John and Andrew come to a river and decideto perform a 200 m swimming contest. The river was $d = 100\,\mathrm{m}$ wide, its flow rate is $u = 0.9\,\mathrm{m}\cdot\mathrm{s}^{-1}$. In still water, both the boys can swim at speeds of $v = 1.5\,\mathrm{m}\cdot\mathrm{s}^{-1}$. Andrew decides to swim across the river and back, John prefers to swim 100 m upstream and back. Who is the winner?
Hint: Velocity vector of each swimmer is the sum of his velocity relative to water and the velocity of water. Draw a picture.

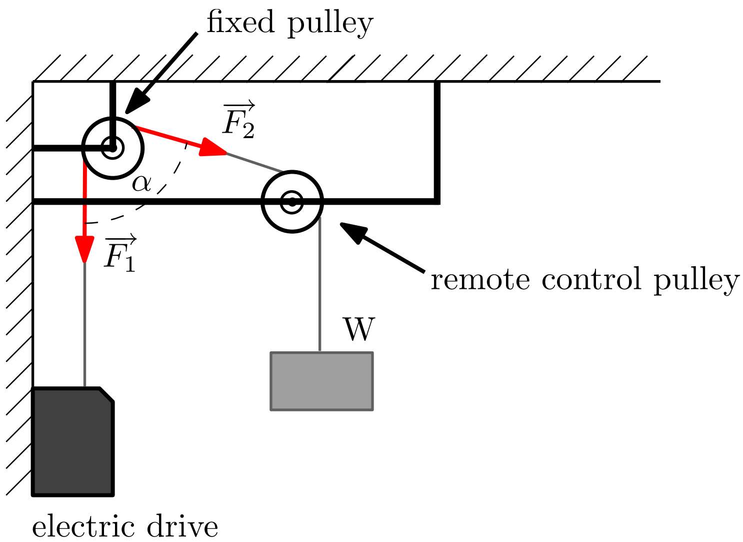


Figure 2.12: *The pulley system used in the factory from problem 2.5*

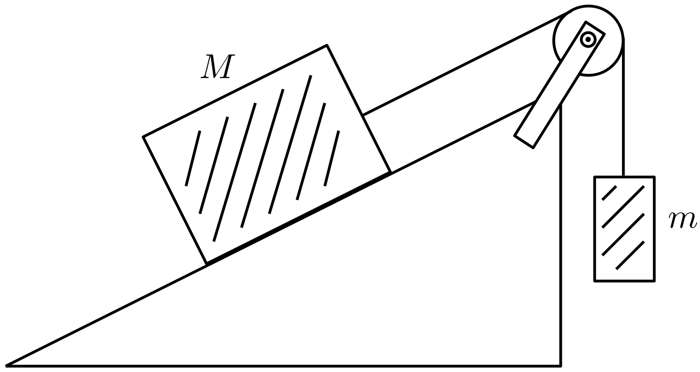


Figure 2.13: *The block and weight from problem 2.8*

2.5 Multiplying Vectors – the Scalar and Vector Products

The **scalar (dot) product** of the vectors $\boldsymbol{A}$ and $\boldsymbol{B}$ is defined to be the product of the length of $\boldsymbol{A}$ times the length of $\boldsymbol{B}$ times cosine of the angle φ between them.

- It follows from figure 2.14 that it is also the product of the length of $\boldsymbol{A}$ projected onto $\boldsymbol{B}$ times the length of $\boldsymbol{B}$, or vice versa. If the angle φ between two vectors $\boldsymbol{A}$ and $\boldsymbol{B}$ is 90°, their scalar product is zero.

$$\boldsymbol{A} \cdot \boldsymbol{B} = |\boldsymbol{A}|\,|\boldsymbol{B}| \cos\varphi = A\,B \cos\varphi$$

- The scalar product of the two vectors is a scalar quantity.

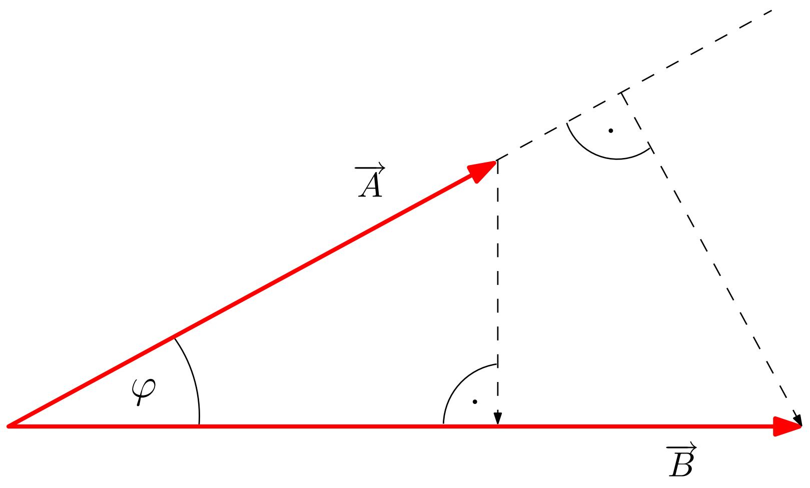


Figure 2.14: *Scalar product of vectors* $\boldsymbol{A}$ *and* $\boldsymbol{B}$

- The commutative law applies

$$\boldsymbol{A} \cdot \boldsymbol{B} = \boldsymbol{B} \cdot \boldsymbol{A}.$$

- The distributive law applies

$$(\boldsymbol{A} + \boldsymbol{B}) \cdot (\boldsymbol{C} + \boldsymbol{D}) = \boldsymbol{A} \cdot \boldsymbol{C} + \boldsymbol{A} \cdot \boldsymbol{D} + \boldsymbol{B} \cdot \boldsymbol{C} + \boldsymbol{B} \cdot \boldsymbol{D}.$$

- Note that $\boldsymbol{i} \cdot \boldsymbol{i} = \boldsymbol{j} \cdot \boldsymbol{j} = \boldsymbol{k} \cdot \boldsymbol{k} = 1$, scalar products of different base vectors are zero.

- Once we choose a coordinate system with base vectors $\boldsymbol{i}$, $\boldsymbol{j}$, and $\boldsymbol{k}$, the vectors $\boldsymbol{A} = A_x\boldsymbol{i} + A_y\boldsymbol{j} + A_z\boldsymbol{k}$ and $\boldsymbol{B} = B_x\boldsymbol{i} + B_y\boldsymbol{j} + B_z\boldsymbol{k}$, it follows according to the distributive law

$$\boldsymbol{A} \cdot \boldsymbol{B} = A_xB_x + A_yB_y + A_zB_z$$

Sample problem 2.7 Giant swings are favourite adrenaline attractions where you can experience overload up to $2G$. A swing can be composed of two equal ropes attached to high pillars ($h_1 = 15\,\mathrm{m}$ each, $d = 10\,\mathrm{m}$ apart) and connected to the jumper's belt ($h_2 = 1\,\mathrm{m}$ above his feet). A jumper gets ready to jump on another pillar, $h_3 = 13\,\mathrm{m}$ high and $l = 12\,\mathrm{m}$ from the middle point between the first two pillars. Choose the coordinate system as in figure 2.15. Determine the position vectors of the fixed ends of the ropes and a jumper, the displacement vectors from the jumper to these ends, the length of the ropes and the angle between them.
Solution:

$$\begin{aligned} \boldsymbol{r}_A &= (0, 0, 14)\,\mathrm{m}, \\ \boldsymbol{r}_B &= (12, -5, 15)\,\mathrm{m}, \\ \boldsymbol{r}_C &= (12, 5, 15)\,\mathrm{m}, \end{aligned}$$

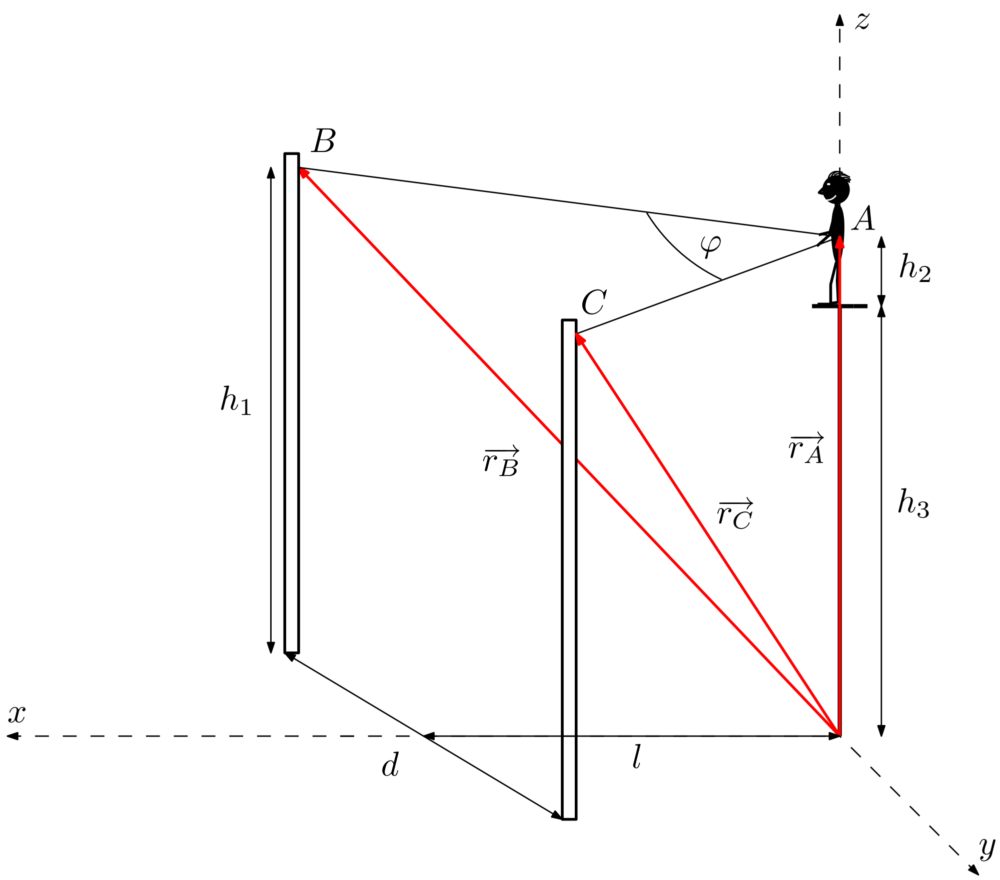


Figure 2.15: *A giant swing*

$$\begin{aligned}
\boldsymbol{r}_{AB} &= \boldsymbol{r}_B - \boldsymbol{r}_A = [(12,-5,15) - (0,0,14)]\,\mathrm{m} = (12,-5,1)\,\mathrm{m}, \\
\boldsymbol{r}_{AC} &= \boldsymbol{r}_C - \boldsymbol{r}_A = [(12,5,15) - (0,0,14)]\,\mathrm{m} = (12,5,1)\,\mathrm{m}, \\
|\boldsymbol{r}_{AB}| = |\boldsymbol{r}_{AC}| &= \sqrt{12^2+5^2+1^2}\,\mathrm{m} = \sqrt{170}\,\mathrm{m} \doteq 13\,\mathrm{m} \\
\cos\varphi &= \frac{\boldsymbol{r}_{AB}\cdot\boldsymbol{r}_{AC}}{|\boldsymbol{r}_{AB}||\boldsymbol{r}_{AC}|} = \frac{12\cdot 12 + 5\cdot(-5) + 1\cdot 1}{\sqrt{170}\cdot\sqrt{170}} = \frac{120}{170} \doteq 0.7 \\
\varphi &\doteq 45^\circ
\end{aligned}$$

! The **vector (cross) product** of the vectors $\boldsymbol{A}$ and $\boldsymbol{B}$ produces a **vector** whose magnitude is the product of the length of $\boldsymbol{A}$ times the length of $\boldsymbol{B}$ times sine of the smaller angle φ between them. It is perpendicular to the plane of multiplied vectors and it obeys the right-hand rule. Shift the vectors $\boldsymbol{A}$ and $\boldsymbol{B}$ tail to tail. Imagine holding your right hand with thumb outstretched thumb in such a way that your pointer points in the direction of $\boldsymbol{A}$ (see Fig. 2.16) and when your hand rotates along the thumb axis in the direction of the open palm, the pointer revolves from $\boldsymbol{A}$ to $\boldsymbol{B}$ through the smaller angle between them. Then the thumb shows the direction of their vector product.

- The vector product of the two vectors is a vector quantity.
- The anticommutative law applies

$$\boldsymbol{A}\times\boldsymbol{B} = -\boldsymbol{B}\times\boldsymbol{A}.$$

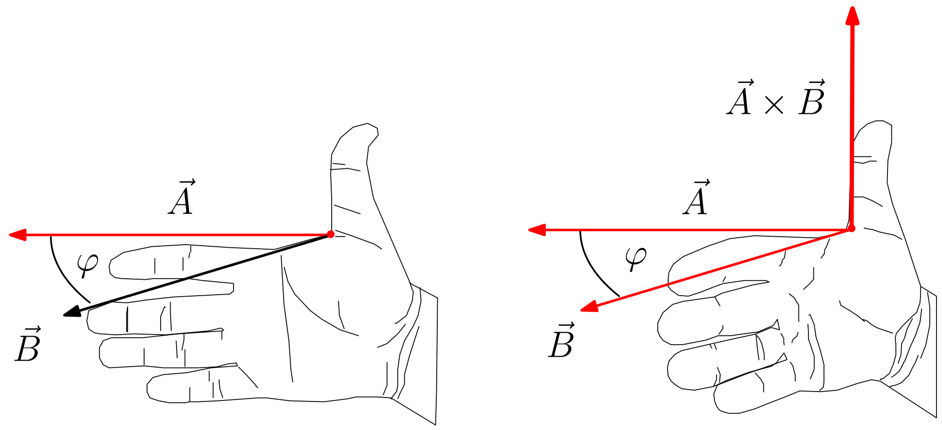


Figure 2.16: *Vector product of vectors* $\boldsymbol{A}$ *and* $\boldsymbol{B}$

- The distributive law applies

$$(\boldsymbol{A}+\boldsymbol{B})\times(\boldsymbol{C}+\boldsymbol{D}) = \boldsymbol{A}\times\boldsymbol{C}+\boldsymbol{A}\times\boldsymbol{D}+\boldsymbol{B}\times\boldsymbol{C}+\boldsymbol{B}\times\boldsymbol{D}.$$

- Note that $\boldsymbol{i}\times\boldsymbol{i}=\boldsymbol{j}\times\boldsymbol{j}=\boldsymbol{k}\times\boldsymbol{k}=\boldsymbol{0}$, $\boldsymbol{i}\times\boldsymbol{j}=\boldsymbol{k}$, $\boldsymbol{j}\times\boldsymbol{k}=\boldsymbol{i}$, $\boldsymbol{k}\times\boldsymbol{i}=\boldsymbol{j}$, $\boldsymbol{j}\times\boldsymbol{i}=-\boldsymbol{k}$, $\boldsymbol{k}\times\boldsymbol{j}=-\boldsymbol{i}$ and $\boldsymbol{i}\times\boldsymbol{k}=-\boldsymbol{j}$.

- Once we choose a coordinate system with base vectors $\boldsymbol{i}$, $\boldsymbol{j}$, and $\boldsymbol{k}$, the vectors $\boldsymbol{A}=A_x\boldsymbol{i}+A_y\boldsymbol{j}+A_z\boldsymbol{k}$ and $\boldsymbol{B}=B_x\boldsymbol{i}+B_y\boldsymbol{j}+B_z\boldsymbol{k}$, it follows according to the distributive law

$$\boldsymbol{A}\times\boldsymbol{B}=\begin{vmatrix}\boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k}\\ A_x & A_y & A_z\\ B_x & B_y & B_z\end{vmatrix}$$

or

$$\begin{aligned}(\boldsymbol{A}\times\boldsymbol{B})_x &= (A_yB_z-A_zB_y)\\ (\boldsymbol{A}\times\boldsymbol{B})_y &= (A_zB_x-A_xB_z)\\ (\boldsymbol{A}\times\boldsymbol{B})_z &= (A_xB_y-A_yB_x)\end{aligned}$$

2.6 Statics of a Rigid Body

A rigid body is a body that can move without any change of shape. In the first chapter we encountered Newton's laws of motion, determining translation motion of a mass point or a rigid body. We have seen that if the resultant force (net force) acting on an object is zero, an object will either move along a straight line at a constant speed, or it will remain motionless (maintain its zero speed). To change its velocity, we need to apply a non-zero net force on an object.

Imagine a rigid bar lying still on an ice rink. Let's suppose we apply a pair of opposite forces, one at each end. The sum of the forces exerted on the bar thus remains zero. Will the bar stay still? Of course not – it will be set into **rotation**. The motion of rotation can be described in much the same way as the motion of translation, using the rotational variables (angular position, angular velocity, angular acceleration, etc.) instead of translational variables. We shall study all of them in more detail later. For now we introduce just a **torque**, which can cause a change in the rotation of a rigid body.

The torque (momentum of force) on a body (relative to some reference point) can be defined as the cross product:

$$\boldsymbol{M} = \boldsymbol{r} \times \boldsymbol{F},$$

where $\boldsymbol{F}$ is the force acting on the body and $\boldsymbol{r}$ is displacement vector from the reference point to the point of action.

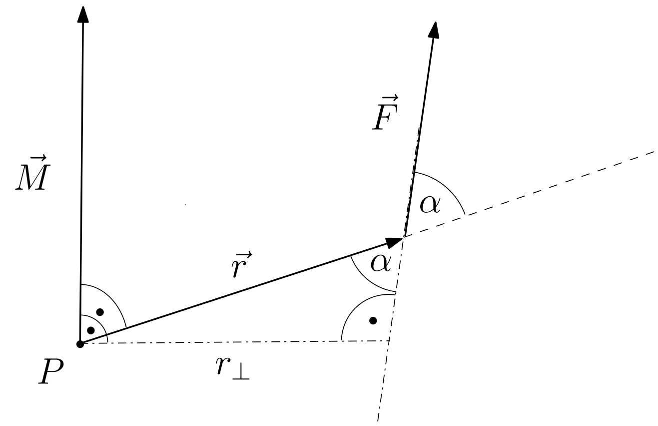


Figure 2.17: *Torque of force $\boldsymbol{F}$ about the point P*

It follows that its magnitude must be

$$M = rF \sin\varphi$$

For a body to be in static equilibrium, not only must the sum of the forces $\boldsymbol{F}_1$, $\boldsymbol{F}_2$, $\cdots$ $\boldsymbol{F}_n$ be zero, but also the sum of their torques around any point.

$$\sum_{i=1}^{n} \boldsymbol{F}_i = \boldsymbol{0},$$

$$\sum_{i=1}^{n} \boldsymbol{M}_i = \boldsymbol{0},$$

where $\boldsymbol{M}_i = \boldsymbol{r}_i \times \boldsymbol{F}_i$ and the origin of the vectors $\boldsymbol{r}_i$ is arbitrary, but the same point on the body or outside of it.

In practice, studying rotation around fixed axes is often useful. Look at the pincers in figure 2.18. The rotational axis of the handles is represented by the point P at the hinge. Note that the magnitude of torque about this point due to the force on each handle can also be calculated as

$$M = Fr_\perp,$$

where $r_\perp$ is the perpendicular distance between the rotation axis at P and the extended line running through the vector $\boldsymbol{F}$, called the **line of action**. This perpendicular distance is called the **moment arm** of the force. It follows from the equilibrium of force momenta that the force F' exerted on each side of an object (e.g. a wire) by the jaws of the pincers then obeys the rule:

$$F'r'_{\perp} = Fr_{\perp}$$

This is the principle of a lever: the equilibrium is established when the sum of the torques acting in a clockwise direction is equal to the sum of the torques acting in an anticlockwise direction. It is possible, as a result, to overcome a very large force at a short distance from the fulcrum with a very small force at a great distance from the fulcrum.

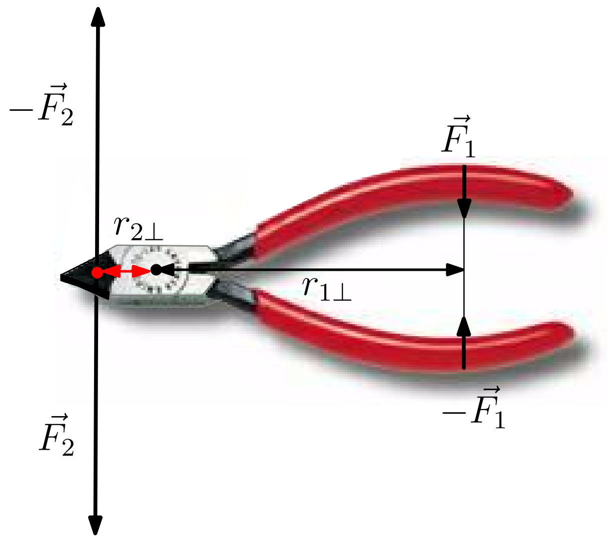


Figure 2.18: *Forces on pincers and their moment arms*

Sample problem 2.8 Determine how to set up the ladder so that a worker can climb it and stand safely at the third rung from the top. The side rails are $L = 9$ m long, the distances between the rungs and from the rungs to the ends of the ladder are $d = 30$ cm. The mass of a worker is $m_1 = 90$ kg, the mass of the ladder $m_2 = 30$ kg, the coefficient of static friction between the ladder and the wall is $\mu_1 = 0.5$, the coefficient of static friction between the ladder and the ground is $\mu_2 = 0.7$. If the ladder leans against the wall, how far from the wall should the feet of the ladder be?

Solution: We draw a picture of the ladder and all the forces exerted on it. They include

- the gravitational force on the ladder $\boldsymbol{G} = m_2\boldsymbol{g}$,
- the weight of the worker $\boldsymbol{W} = m_1\boldsymbol{g}$
- the normal force from the wall surface $\boldsymbol{N}_1$,
- the frictional force between the ladder and the wall opposite to the direction of intended motion (slippage) of the ladder $\boldsymbol{F}_1$.
- the normal force from the ground surface $\boldsymbol{N}_2$,
- the frictional force between the ladder and the ground opposite to the direction of intended motion $\boldsymbol{F}_2$.

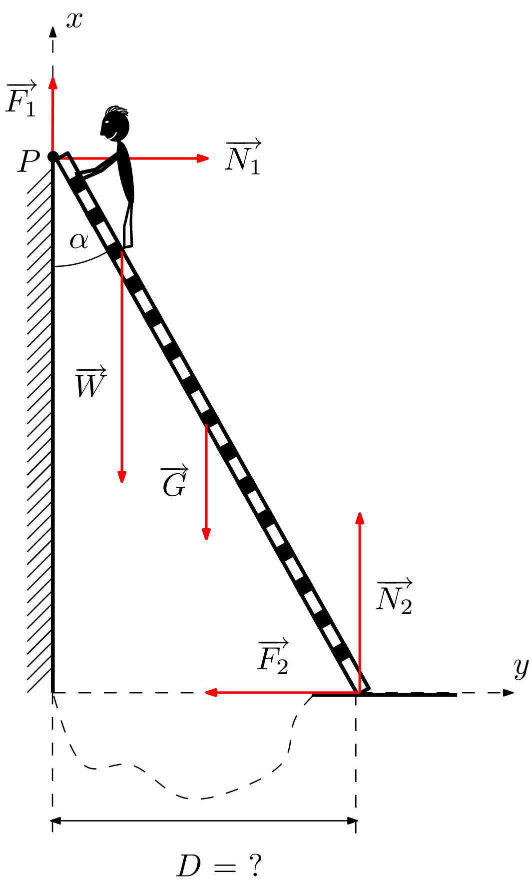


Figure 2.19: *Forces on a ladder leaning against the wall*

For the ladder to be in static equilibrium, the sum of the forces and the sum of their torques around any point must be zero. It follows

$$\boldsymbol{G} + \boldsymbol{W} + \boldsymbol{N}_1 + \boldsymbol{F}_1 + \boldsymbol{N}_2 + \boldsymbol{F}_2 + \boldsymbol{0}$$

and

$$\boldsymbol{M_G} + \boldsymbol{M_W} + \boldsymbol{M_{N1}} + \boldsymbol{M_{F1}} + \boldsymbol{M_{N2}} + \boldsymbol{M_{F2}} + \boldsymbol{0}.$$

To determine the torques, we must first select a reference point. It can be any point on a ladder or elsewhere, suitable selection of this point can simplify the calculation considerably. Let's pick the point P at the top of the ladder. All the forces lie on the $x - y$ plane, choosing a reference point on the same plane implies the torques must point in the direction of z axis or opposite. So the couple of vector equations (generally two times three equations when expressed in a coordinate system) is reduced to just three equations as follows:

$$\begin{aligned} N_2 + F_1 - G - W &= 0 \\ N_1 - F_2 &= 0 \\ W(3d)\sin\alpha + G\frac{L}{2}\sin\alpha - N_2 L \sin(\pi - \alpha) + F_2 L \sin\left(\frac{\pi}{2} + \alpha\right) &= 0 \end{aligned}$$

The last equation can be simplified as

$$W(3d)\sin\alpha + G\frac{L}{2}\sin\alpha - N_2 L \sin\alpha + F_2 L \cos\alpha = 0$$

Two more equations follow from the relation between normal and frictional forces:

$$F_1 = \mu_1 N_1$$
$$F_2 = \mu_2 N_2$$

Solving this system of five equations we obtain the solution

$$\tan\alpha = \frac{\mu_2(m_1+m_2)}{m_1+m_2-(1+\mu_1\mu_2)(3dm_1/L+0.5m_2)},$$

so $\alpha \doteq 44°$ and $D = L\sin\alpha \doteq 6.2$m.

Of course, this is the maximum possible distance - it is recommended to stand a ladder more closely to the wall, but not totally upright. If the vertical from the centre of mass of a worker who starts climbing the ladder intersects the ground in front of the ladder, it could fall over backwards - think this over case on your own. The golden mean is usually recommended.

Sample problem 2.9 A driver attempts to get a car onto the decking driving it up a loose-rigged ramp that rests against a large fixed wooden crate. The car has a rear-axle drive, neglecting the friction in its front axle, we can suppose the tyres create force perpendicular to the ramp, its magnitude is $F = 1000$ N per each front wheel. The length of the ramp is $l = 3$ m. The distance from the lower end of the ramp to the front wheels is $a = 1$ m. The ramp is elevated at a $\alpha = 20°$ angle. The coefficient of static friction between the ramp and the crate is $\mu_1 = 0.2$. At this moment, the car is in a position as in figure 2.20. What is the minimum value for the coefficient of static friction between the ramp and the ground μ_2 that will prevent the ramp from moving? Suppose the weight of the ramp is negligible.

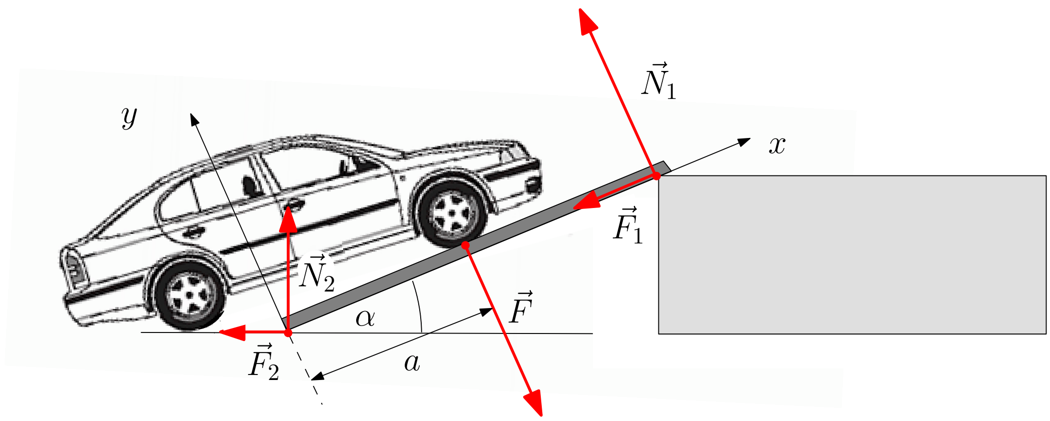


Figure 2.20: *Forces on the ramp holding the front-axle of a car (not to scale)*

Solution: First, we draw a free-body diagram of the ramp with the friction acting against its intended motion - to the left. The forces on the ramp include:

- the normal force on the ramp from each front wheel $\boldsymbol{F}$,
- the normal force from the crate surface $\boldsymbol{N}_1$,

- the frictional force between the ramp and the crate $\boldsymbol{F}_1$.
- the normal force from the ground surface $\boldsymbol{N}_2$,
- the frictional force between the ramp and the ground $\boldsymbol{F}_2$.

We also use the relations between friction and normal forces and the equilibrium equations for forces and their momenta (torques). We shall set our reference point at the lower end of the ramp. Using a coordinate system, we obtain:

$$\begin{aligned} N_2 \sin\alpha - F_2 \cos\alpha - F_1 &= 0 \\ N_2 \cos\alpha + F_2 \sin\alpha - 2F + N_1 &= 0 \\ 2Fa - N_1 l &= 0 \\ F_2 &= \mu_1 N_1 \\ F_1 &= \mu_2 N_2 \end{aligned}$$

This system of equations yields the solution

$$\mu_2 = \frac{(l-a)\sin\alpha - \mu_1 a \cos\alpha}{(l-a)\cos\alpha + \mu_1 a \sin\alpha} = \tan\alpha \, \frac{l - a(1 + \mu_1 \cot\alpha)}{l - a(1 - \mu_1 \tan\alpha)} = 0.25.$$

This is the lowest value of μ_2 needed to prevent motion. Note that the result does not depend on the normal force exerted on the ramp by the car. This force would play a role if the weight of the ramp were impossible to disregard. The most critical position is when $a = 0$, then $\mu_2 = \tan\alpha = 0.36$.

Problem 2.10 Given the two vectors $\boldsymbol{A} = (1,2,3)$, $\boldsymbol{B} = (3,2,1)$, find a) their scalar product, b) their vector product and c) the angle between them.

Problem 2.11 If you know that $\boldsymbol{A} \cdot \boldsymbol{B} = 20$ and $\boldsymbol{A} \times \boldsymbol{B} = (0,0,20)$, are $\boldsymbol{A}$ and $\boldsymbol{B}$ uniquely determined?

Problem 2.12 Consider once more a situation from the sample problem with a giant swing (figure 2.15). Our jumper swung down like a pendulum and the net force he exerted on the pair of ropes at his lowest point was $F = 2100\,\mathrm{N}$. Using the same data and the same coordinate system as in the sample problem, determine the vectors of reaction tension forces from each rope and their magnitudes.

Problem 2.13 In figure 2.21 you can see two children enjoying themselves on a see-saw. It is usually constructed with the fulcrum in the middle. If the distance between the seats is $l = 3\,\mathrm{m}$, the mass of the first child $m_1 = 25\,\mathrm{kg}$ and the mass of the second $m_2 = 50\,\mathrm{kg}$, how far from the first child would the optimal position of the fulcrum be to establish the equilibrium of the torques? Does it depend on the momentary inclination of the see-saw?

Problem 2.14 (This exercise is optional): A car climbs an inclined road, as in figure 2.22. The road is elevated at an $\alpha = 20^\circ$ angle. The total weight of the car is $W = 13.3\,\mathrm{kN}$, its wheel base $l = 2.5\,\mathrm{m}$ and its centre of mass is in the middle at $h = 0.7\,\mathrm{m}$ above the road. Determine normal forces from the road on the front wheels and the rear wheels.

Figure 2.21: *A see-saw, problem 2.13*

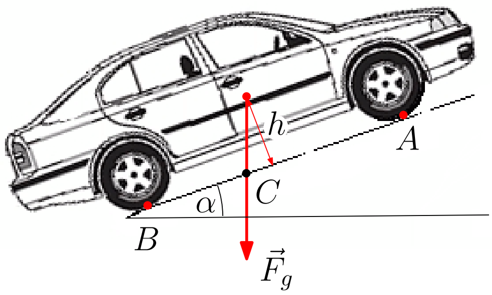


Figure 2.22: *A car on a ramp, problem 2.14*

Determine the minimum coefficient of static friction which prevents the car from sliding down if it has
a) front-wheel drive,
b) rear-wheel drive,
c) four-wheel drive.

Chapter 3

Motion and the Mathematics of Diminutive Changes

?

How to determine the instantaneous speed of a car which accelerates?

How can mathematics deal with infinitesimal changes? What are the basic rules of such calculus?

Why does physics need infinitesimal time intervals?

Can the mathematics of diminutive changes be useful somewhere else?

After finishing this chapter you should be able to

- define average and instantaneous speed and acceleration,
- differentiate some simple functions,
- calculate average and instantaneous speed, if the distance is known as a function of time,
- calculate average and instatnaneous acceleration, if the speed is known as a function of time,
- evaluate an indefinite integral of some simple functions, including the integrals solvable using the per partes method or simple substitutions,
- calculate the definite integral of these functions,
- calculate the distance if the speed is known as a function of time
- calculate the speed if the acceleration is known as a function of time.

3.1 Motion in One Direction – Instantaneous Speed and Acceleration

A Ferrari roadster, originally at rest, speeds up on a straight motorway and travels a distance of 400 m in 12.7 s. Its distance from the start point grows with time as in figure 3.1. Determine its speed.

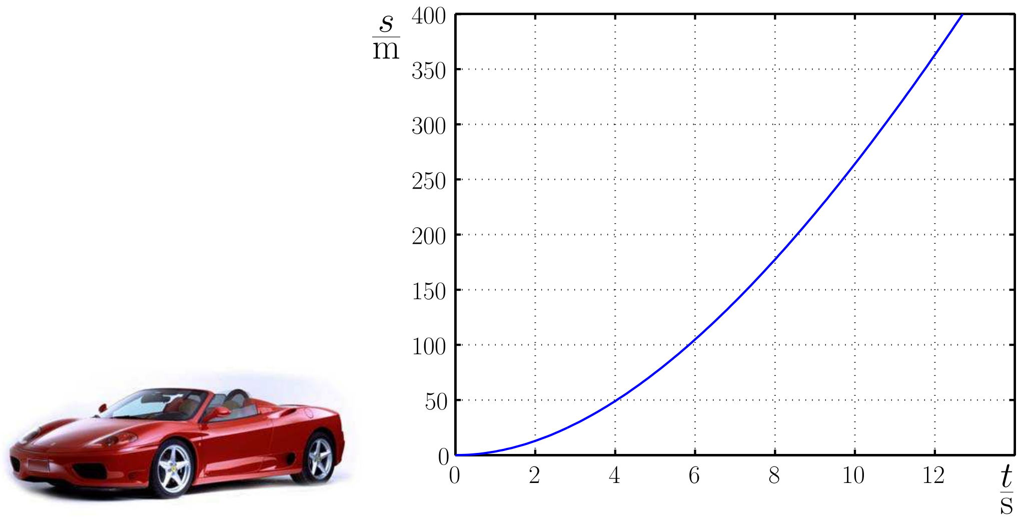


Figure 3.1: *The Ferrari and its distance from the start as a function of time*

We can easily calculate the **average speed** of the car during the time interval, which is the ratio of the total distance and the total time interval over which the displacement takes place:

$$v = \frac{s}{t} = \frac{400\ \mathrm{m}}{12.7\ \mathrm{s}} = 31,5\ \mathrm{m.s^{-1}} = 113.4\ \mathrm{km.h^{-1}}$$

However, this information is not very valuable when you are interested in its speed at a particular moment. As we know, at the beginning the speed was zero, the final speed would be much higher than the average one. How can we find the instantaneous velocity, the speed that a Ferrari driver could see on his speedometer? Surely we could obtain better result if we take into account not the whole time of 12.7 s, but a shorter period of time from the selected point. We get the average speed of the car over a shorter period of time as the quotient of the distance Δs and the time interval Δt over which it takes place. It becomes apparent that shrinking the time interval yields the more accurate result.

$$v = \frac{\Delta s}{\Delta t}$$

We therefore shrink the time interval close to zero (take an **infinitesimal** time interval).

The **instantaneous speed** (or simply speed) is the limit of the ratio of the distance and the time interval for the time interval approaching a zero value:

$$v = \lim_{\Delta t \to 0} \frac{\Delta s}{\Delta t} = \frac{\mathrm{d}s}{\mathrm{d}t}$$

Such a limit is called a differentiation of s with respect to t. Thus we obtain the value of an instantaneous speed at any time, and we can draw the speed as a function of time, as in figure 3.4.

In much the same way, average acceleration is defined as the quotient of the change in speed Δv and the time interval Δt over which the change in speed takes place:

$$a = \frac{\Delta v}{\Delta t}$$

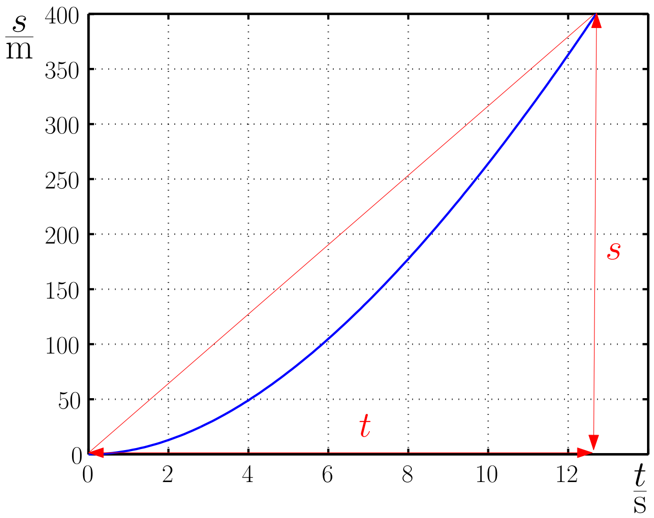


Figure 3.2: *Determination of the average velocity during the whole interval*

The **instantaneous acceleration** at time t is defined as the limit of the average acceleration as Δt approaches zero.

$$a = \lim_{\Delta t \to 0} \frac{\Delta v}{\Delta t} = \frac{\mathrm{d}v}{\mathrm{d}t}$$

3.2 Introduction of Derivatives

Before we make use of derivatives in physical applications, let's study them in more detail. First of all, we must emphasise that the following text is by no means intended as a rigorous development of differential calculus theory, this is left to your courses in mathematics. We only attempt here to supply a basic insight and some rules necessary for simple physical applications.

Let $f(x)$ be a function of variable x. The notation of its derivative (differentiation) with respect to x is $f'(x)$. A derivative represents the rate of change of a function and a variable. The derivative of a function $f(x)$ at a given point x is defined as:

$$f'(x) = \frac{\mathrm{d}f(x)}{\mathrm{d}x} = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

We assume that the limit exists, although it needn't always be the case. We can also define the second derivative of $f(x)$ with respect to x as

$$f''(x) = \frac{\mathrm{d}}{\mathrm{d}x}\left[\frac{\mathrm{d}\,f(x)}{\mathrm{d}x}\right] = \frac{\mathrm{d}^2 f(x)}{\mathrm{d}x^2}.$$

Actually we have met it already defining instantaneous acceleration – it is a second derivative of distance with respect to time. Look at figure 3.5. Note that the tangent line to $y = f(x)$ at point $(x, f(x))$ is the line that passes through this point and whose slope is

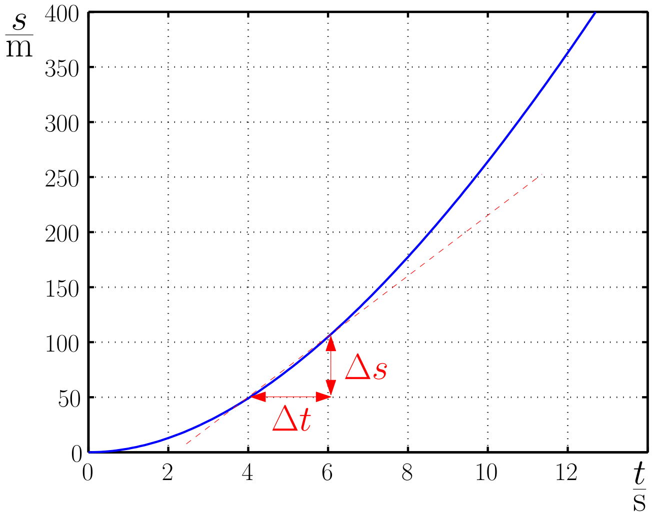


Figure 3.3: *Determination of the average velocity during a short time interval*

equal to the derivative of $f'(x)$,

$$\tan\alpha = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h} = f'(x).$$

It follows that at some point x_0 the function $f(x)$ is

- increasing (ascending) if $f'(x_0) > 0$,
- decreasing (descending) if $f'(x_0) < 0$,
- stationary if $f'(x_0) = 0$. In this case it has
  - a local maximum at x_0 if $f''(x_0) < 0$,
  - a local minimum at x_0 if $f''(x_0) > 0$,
  - an inflection point at x_0 if $f''(x_0) = 0$.

Calculating derivatives according to the definition is tedious. Derivative formulae of basic functions already found can be used to get the result without having to use the definition directly. Here we present just a few of them:

1. If C=const. $\Rightarrow$ $[C]' = 0$
2. If $n \neq 0$ and $x > 0$ $\Rightarrow$ $[x^n]' = n\,x^{n-1}$
3. $[\mathrm{e}^x]' = \mathrm{e}^x$
4. $[\sin\,x]' = \cos\,x$
5. $[\cos\,x]' = -\sin\,x$

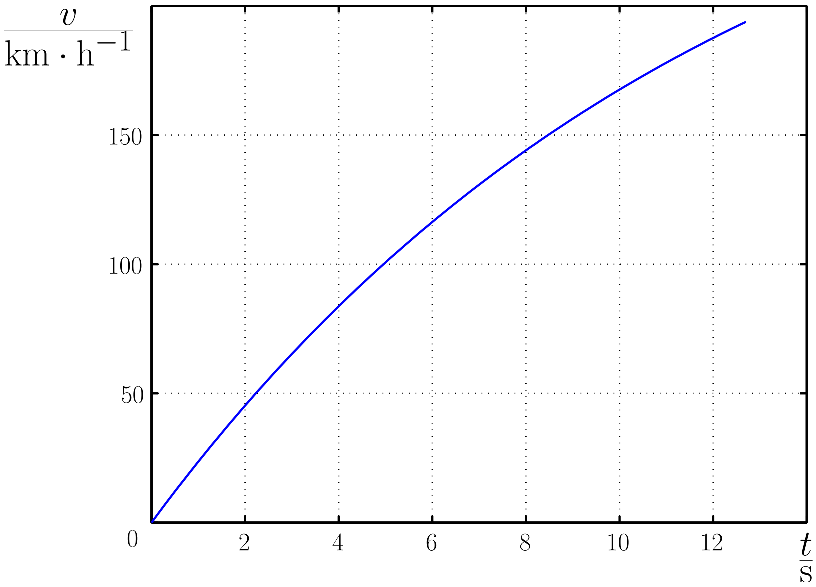


Figure 3.4: *The Ferrari's speed as a function of time*

6. If $x \neq 0 \quad \Rightarrow \ [\ln x]' = \frac{1}{x}$

   Some simple rules have been developed for finding derivatives of some combinations of functions; let's have the functions $f(x)$ and $g(x)$. It holds

7. $[f(x) + g(x)]' = f'(x) + g'(x)$

8. $[c\,f(x)]' = c\,f'(x)$

9. $[f(x)\,g(x)]' = f'(x)\,g(x) + f(x)\,g'(x)$

10. $\left[\frac{f(x)}{g(x)}\right]' = \frac{f'(x)\,g(x) - f(x)\,g'(x)}{[g(x)]^2}$

    Now consider one function being the argument of another function. It holds:

11. $\frac{\mathrm{d}}{\mathrm{d}x} f\big(g(x)\big) = \frac{\mathrm{d}f(g)}{\mathrm{d}g} \cdot \frac{\mathrm{d}g(x)}{\mathrm{d}x}$

Sample problem 3.1 $y = \ln\left(2x + \sqrt{x}\right)$. Find y'.
Solution: There is a function $f(x) = (2x + \sqrt{x})$ that is an argument of the function $y\,(f(x)) = \ln\left(f(x)\right)$. It is the summation of two other functions: $f(x) = g(x) + h(x)$, where $g(x) = 2x$ and $h(x) = \sqrt{x}$.

- If we differentiate y with respect to x, we first apply the rule 11:

$$y' = \frac{\mathrm{d}y(f)}{\mathrm{d}f} \cdot \frac{\mathrm{d}f(x)}{\mathrm{d}x} = \frac{\mathrm{d}\ln(f)}{\mathrm{d}f} \cdot \frac{\mathrm{d}f(x)}{\mathrm{d}x} = \frac{1}{f} \cdot \frac{\mathrm{d}f(x)}{\mathrm{d}x}.$$

- Then we apply rule 7, which implies

$$\frac{\mathrm{d}f(x)}{\mathrm{d}x} = f'(x) = [g(x) + h(x)]' = g'(x) + h'(x).$$

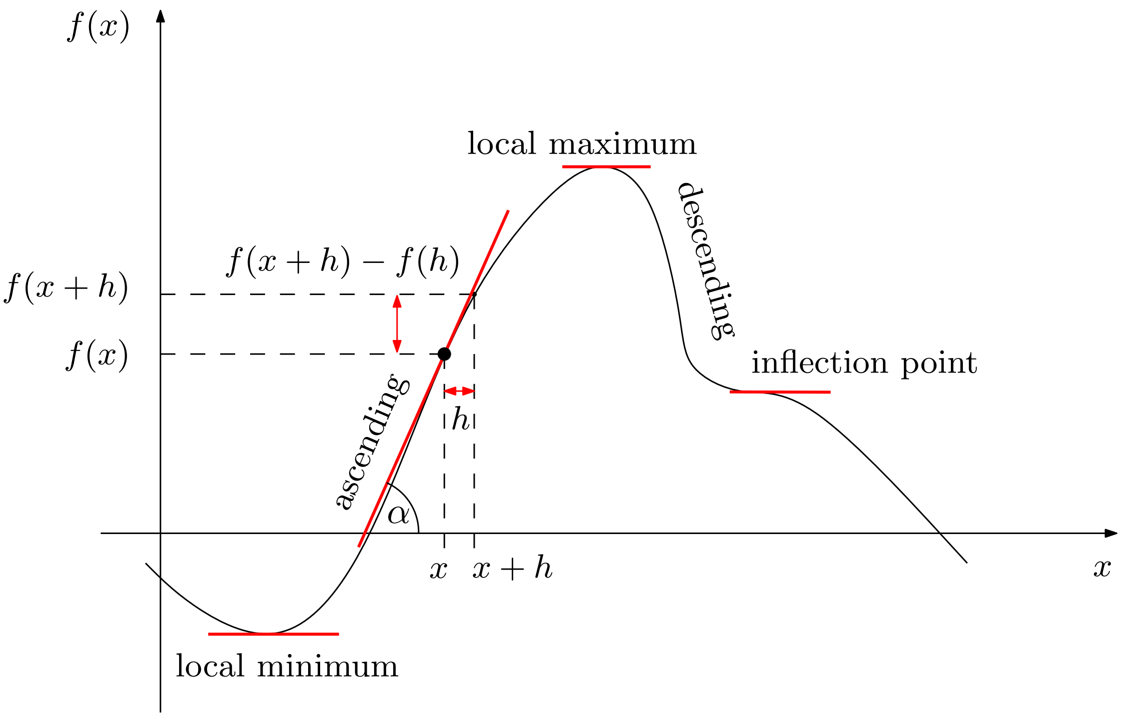


Figure 3.5: *Some geometrical attributes of derivatives*

- To find

$$g'(x) = [2x]' = 2$$

we have used rules 8 and 2, the latter of which also implies

$$h'(x) = [\sqrt{x}]' = [x^{\frac{1}{2}}]' = \frac{1}{2} \cdot x^{-\frac{1}{2}} = \frac{1}{2\sqrt{x}}.$$

- Finally, we obtain

$$y' = \frac{1}{f(x)} \cdot \frac{\mathrm{d}f(x)}{\mathrm{d}x} = \frac{1}{2x + \sqrt{x}} \left(2 + \frac{1}{2\sqrt{x}}\right) = \frac{2 + \frac{1}{2\sqrt{x}}}{2x + \sqrt{x}} = \frac{4\sqrt{x} + 1}{2x\,(2\sqrt{x} + 1)}.$$

Sample problem 3.2 Fermat's principle states that the path of a ray of light between two points is the path that minimises the travel time. Task: using this principle find the Snell's law of refraction for the light crossing the boundary of two media with indices of refraction n_1 and n_2 (a refraction index determines the ratio of the speed of light in vacuum to the speed of light in a given medium).
Solution: As the speed of light doesn't change with its position in a homogeneous medium, a light ray will follow a straight line here. It is not the case if the speed of light varies in space. To connect two points in media with different indices of refraction, it would be more efficient to travel a longer distance through a medium where the speed of light is higher. Figure 3.6 shows several candidate rays for refraction. They differ in the points of intersection of rays at the boundary of the two media. The position of an intersection point can be represented by the parameter x (as in figure 3.7 on the right).

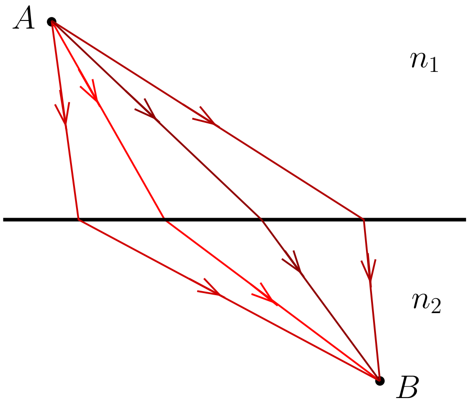


Figure 3.6: *Possible routes for a ray of light connecting two points*

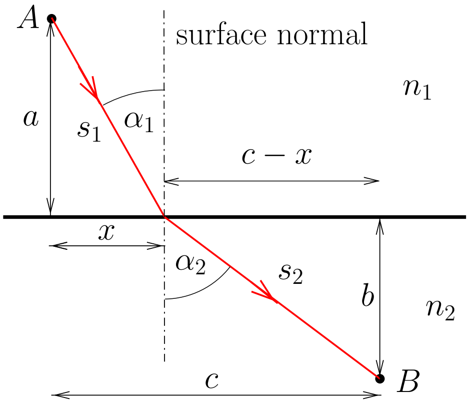


Figure 3.7: *Finding the quickest route for a ray of light connecting two points*

The time required for the light to go from point A to point B is

$$t = \frac{n_1 s_1}{v} + \frac{n_2 s_2}{v} = \frac{n_1\sqrt{a^2 + x^2}}{v} + \frac{n_2\sqrt{b^2 + (c-x)^2}}{v},$$

where v is the speed of light in vacuum. We find the minimum time by differentiating t with respect to x and setting the result to zero:

$$\frac{\mathrm{d}t}{\mathrm{d}x} = \frac{n_1}{v}\frac{x}{\sqrt{a^2 + x^2}} - \frac{n_2}{v}\frac{c-x}{\sqrt{b^2 + (c-x)^2}} = 0$$

However, we note that

$$\frac{x}{\sqrt{a^2 + x^2}} = \sin\alpha_1$$

and

$$\frac{c-x}{\sqrt{b^2 + (c-x)^2}} = \sin\alpha_2$$

consequently the minimum time condition reduces to

$$n_1 \ \sin\alpha_1 = n_2 \ \sin\alpha_2,$$

which is the well-known Snell's law of refraction.

Problem 3.1 Differentiate the following functions in square brackets with respect to x:

(a) $[x^2 + 2x - 3]'$

(b) $\left[\frac{1}{x} + \frac{1}{x^2}\right]'$

(c) $[\sqrt{x}]'$

(d) $\left[\frac{1}{\sqrt{x}}\right]'$

(e) $[\sin 2x]'$

(f) $[\cos^2 x]'$

(g) $\left[\frac{1}{\sin x}\right]'$

(h) $[\tan x]'$

(i) $\left[\sqrt{x^2+1}\right]'$

(j) $\left[\ln \frac{1}{x}\right]'$

(k) $[\sin(2x-3)]'$

(l) $[\mathrm{e}^{1-x}]'$

Problem 3.2 A car is following a straight road when the driver notices a red light on a traffic light. As he starts to brake smoothly, his distance from the lights descends according to a formula: $d = 120 - 20t + t^2$. Has he managed to stop in front of the stop-lights? What is his speed at $t = 2$ s? Determine the deceleration of the car.

Problem 3.3 A skydiver with the total mass $m = 100\,\mathrm{kg}$ jumps out of the plane (without opening his parachute) and his speed grows as

$$v = [57.7 \tanh(0.17t)]\,\mathrm{m \cdot s^{-1}} = \left(57.7\,\frac{\mathrm{e}^{0.17t} - \mathrm{e}^{-0.17t}}{\mathrm{e}^{0.17t} + \mathrm{e}^{-0.17t}}\right)\mathrm{m \cdot s^{-1}}$$

Find his terminal (maximum) speed. Determine his acceleration as a function of time.

Problem 3.4 A weight oscillates on a spring in such a way that its velocity obeys a formula

$$v = 20\mathrm{e}^{-\frac{t}{2}} \sin \frac{t}{2}.$$

When does its speed reach its maximum? Find the value of the maximum speed.

Problem 3.5 Differentiate the function

$$\mu_2 = \tan\alpha\,\frac{l - a(1 + \mu_1 \cot\alpha)}{l - a(1 - \mu_1 \tan\alpha)}$$

(the result of sample problem 2.9) with respect to a and decide if it is for $a \in \langle 0,\, l\rangle$ descending, ascending or it changes its behaviour.

3.3 Displacement of a Variably Accelerated Motion – Introduction of Integrals

Sample problem 3.3 A racing car driver tests his car on the straight part of a race track in such a way that the speed of the car obeys a formula

$$v = 40 + 20 \sin \frac{t}{2}.$$

What was the distance he completed from $t_1 = 0$ s to $t_2 = 10$ s?

To find the displacement of the car, we cannot use the formula $s = v \cdot t$, as it holds for steady motion only, nor the formula $s = v_0 t + \frac{1}{2} a t^2$, which is only valid for constant acceleration. We can easily check that in our case the acceleration varies:

$$a = \frac{\mathrm{d}v}{\mathrm{d}t} = \frac{\mathrm{d}\left(40 + 20 \sin \frac{t}{2}\right)}{\mathrm{d}t} = 10 \cos \frac{t}{2}.$$

It is now the right time to introduce the operation of integration, which is the inverse operation of differentiation. Integration makes it possible to determine a cummulative effect of a variable quantity.

Well, how can we, in full generality, calculate the total displacement s during the time interval (t_1, t_2), if we know velocity as a function of time $v(t)$?

The equation $s = vt$ holds, if $v = \text{const}$. So we first divide our time interval into n small subintervals $(t_i, t_i + \Delta t_i)$ where we assume the velocity to be approximately constant, e.g. $v_i = v(t_i)$. Then $s_i = v_i \Delta t_i$ and the total displacement

$$s \doteq \sum_{i=1}^{n} v_i \ \Delta t_i.$$

To get a more accurate result, we then try to refine our partition. By making better and better approximations (shorter and shorter time subintervals), we can say that, at its limit, we get exactly the area between the velocity curve $v = v(t)$ and time axis, from $t = t_1$ to $t = t_2$. Assuming infinitesimal time intervals $\Delta t_i \rightarrow \mathrm{d}t$, summation turns into integration $s = \int_{t_1}^{t_2} v \ \mathrm{d}t$.

! Displacement s during the time interval (t_1, t_2) is the integration of velocity $v(t)$ over the time

$$s = \int_{t_1}^{t_2} v \ \mathrm{d}t$$

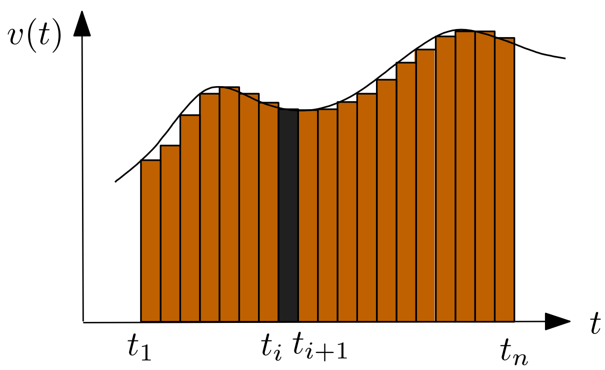


Figure 3.8: *Determination of the total displacement as a sum of distances moved over short time intervals*

Remark: More accurately, the upper sum (approximating the function values with subinterval suprema) and the lower sum (approximating the function values with infima) are considered. The lower sum is always less than or equal to the upper sum, and the upper sum is decreasing with respect to a refinement of the partition while the lower sum is increasing with respect to a refinement of the partition. Finally, we consider time subintervals that are infinitely small. If the upper sum and the lower sum coincide, their value is called the Riemann **definite integral** of $v(t)$ over the interval (t_1, t_2) and it equals the area under the curve we looked for.

In the same way, we can find the velocity attained at a particular time if we know the acceleration. Suppose an object is at rest and it starts to accelerate at t_1.

! If the original velocity is zero, the velocity v attained during the time interval (t_1, t_2) is the integration of acceleration a over that time

$$v = \int_{t_1}^{t_2} a \ \mathrm{d}t.$$

To calculate an integral, we make use of the fact that integration is the inverse operation of differentiation. We now introduce the **indefinite integral**.

An **antiderivative** $F(x)$ of a function $f(x)$ is any function whose derivative is $f(x)$.

$$F'(x) = \frac{\mathrm{d}\, F(x)}{\mathrm{d}x} = f(x).$$

The antiderivative of a function is determined up to an additive constant C, as the derivative of a constant is zero.

We call the set of all antiderivatives of a function the **indefinite integral** of the function. We write the indefinite integral of $f(x)$ with respect to x as

$$\int f(x)\mathrm{d}x.$$

The endpoints of an indefinite integral are not specified. If we add the limits to the integral, the ambiguity in additive constant is eliminated and we obtain the definite integral with the value equal to the area under the curve.

! If $F(x)$ is an antiderivative of $f(x)$ at the interval $\langle a,\, b\rangle$, then the **definite integral** of $f(x)$ from a to b can be calculated as:

$$\int_a^b f(x)\mathrm{d}x = [F(x)]_a^b = F(b) - F(a).$$

Some common integrals are:

1. $\displaystyle\int x^n \mathrm{d}x = \frac{x^{n+1}}{n+1} + C \qquad (x > 0,\ n \neq -1)$

2. $\displaystyle\int \frac{1}{x}\, \mathrm{d}x = \ln|x| + C$

3. $\displaystyle\int \mathrm{e}^x \mathrm{d}x = \mathrm{e}^x + C \quad (x \neq 0)$

4. $\displaystyle\int \sin x \, \mathrm{d}x = -\cos x + C$

5. $\int \cos x \, \mathrm{d}x = \sin x + C$

6. $\int \frac{\mathrm{d}x}{1+x^2} = \arctan x + C$

7. $\int \frac{\mathrm{d}x}{\sqrt{1-x^2}} = \arcsin x + C \qquad x \in < -1, 1 >$

   Some useful rules for calculating integrals are valid. Let's have the functions $f(x)$ and $g(x)$, and a constant c. It holds

8. $\int [f(x) \pm g(x)] \mathrm{d}x = \int f(x) \mathrm{d}x \pm \int g(x) \mathrm{d}x,$

9. $\int c \; f(x) \mathrm{d}x = c \int f(x) \mathrm{d}x$

10. A method of substitution:

$$\begin{aligned} \int f\left[g(x)\right] g'(x) \mathrm{d}x &= \int f(z) \mathrm{d}z \\ \text{substitution:} \quad g(x) &= z \\ g'(x) \mathrm{d}x &= \mathrm{d}z \end{aligned}$$

11. Integration by parts (per partes method):

$$\int f(x) g'(x) \mathrm{d}x = f(x) g(x) - \int f'(x) g(x) \mathrm{d}x$$

There are many more complex methods of dealing with integrals, but this is beyond the scope of the simple introduction, intended by this course.

Solution: (to sample problem 3.3 at the beginning of this section:) The distance can be found as a definite integral

$$\begin{aligned} s &= \int_{t_1}^{t_2} v \; \mathrm{d}t = \int_0^{10} \left(40 + 20 \sin \frac{t}{2}\right) \mathrm{d}t \\ &= \int_0^{10} 40 \mathrm{d}t + \int_0^{10} 20 \sin \frac{t}{2} \mathrm{d}t = 40 \int_0^{10} \mathrm{d}t + 40 \int_0^5 \sin z \mathrm{d}z \\ &= [40t]_0^{10} + [-40 \cos z]_0^5 = (40 \cdot 10 - 40 \cdot 0) + (-40 \cos 5 + 40 \cos 0) = 429 \, \mathrm{m}, \end{aligned}$$

where we have used the rules 8, 9, 1 and substitution $z = t/2$, $\mathrm{d}z = \mathrm{d}t/2$. If we solve the definite integral using substitution, we must not forget to change the limits, so that they are valid for a new variable. In our case $z_1 = t_1/2 = 0/2 = 0$ and $z_2 = t_2/2 = 10/2 = 5$. The other possibility is simply to label the limits formally (e.g. z_1 and z_2) and at the end convert the result to the original variable and use the original limits.

Problem 3.6 Evaluate the following integrals:

(a) $\displaystyle\int \left(x^2 + 2x - 3\right) \mathrm{d}x$

(b) $\displaystyle\int \left(\frac{1}{x} + \frac{1}{x^2}\right) \mathrm{d}x$

(c) $\displaystyle\int \sqrt{x}\, \mathrm{d}x$

(d) $\displaystyle\int \frac{1}{2x-1}\, \mathrm{d}x$

(e) $\displaystyle\int \sin 2x\, \mathrm{d}x$

(f) $\displaystyle\int \cos^2 x\, \mathrm{d}x$

(g) $\displaystyle\int \mathrm{e}^{1-x}\, \mathrm{d}x$

(h) $\displaystyle\int \tan x\, \mathrm{d}x$

(i) $\displaystyle\int x\sqrt{x^2+1}\mathrm{d}x$

(j) $\displaystyle\int x^3 \ln x\, \mathrm{d}x, \quad x > 0$

(k) $\displaystyle\int x \sin x\, \mathrm{d}x$

(l) $\displaystyle\int \mathrm{e}^x \cos x\, \mathrm{d}x$

Problem 3.7 Solve a definite integral: $\displaystyle\int_0^1 \frac{x}{\sqrt{x^2+1}}\mathrm{d}x.$

Problem 3.8 Solve a definite integral: $\displaystyle\int_0^{\pi/2} \sin^2 x \cos x\, \mathrm{d}x.$

Problem 3.9 Remember problem 3.3. A skydiver with the total mass $m = 100\,\mathrm{kg}$ jumped out of the plane (without opening his parachute) and his speed grew as

$$v = [57.7 \tanh(0.17t)]\,\mathrm{m}\cdot\mathrm{s}^{-1} = \left(57.7\,\frac{\mathrm{e}^{0.17t} - \mathrm{e}^{-0.17t}}{\mathrm{e}^{0.17t} + \mathrm{e}^{-0.17t}}\right)\mathrm{m}\cdot\mathrm{s}^{-1}.$$

Find how his distance from the initial point depends on time.

Chapter 4

Planar Motion

Why does a car easily lose control on bends in wet or icy road conditions?

Why are race tracks inclined on the curves?

What is the path of a projected stone?

After finishing this chapter you should be able to

- express a single implicit equation of a circle with the centre at the origin of the coordinate system, write its parametric equations in Cartesian coordinates,
- use a polar coordinate system to determine the position of a point, convert between polar and Cartesian coordinates,
- express a single implicit equation of parabola (with an axis parallel to one of the coordinate axes) and its parametric equations in Cartesian coordinates, read the vertex and the direction from the equation,
- find the intersection of a parabola and a line,
- express the angular displacement, angular velocity and angular acceleration of a particle going around a circle, and show their relation to its displacement, velocity and acceleration,
- find the tangential and centripetal parts of an acceleration, interpret their effects on the motion of a particle
- solve simple problems concerning uniform and non-uniform circular motion,
- solve simple problems concerning motion in the homogeneous gravitational field of the Earth.

4.1 Circular Motion

In the last chapter (3), we studied the motion of a particle along a straight line. This may be too restrictive, as in practice we often experience motion in two or three dimensions.

In this chapter, we shall investigate **curvilinear motion**, motion where an object follows a curve. In this chapter, we will only pay attention to motion in two dimensions.

A very simple example is uniform circular motion, as may be the motion of a satellite traveling around the Earth or a car turning at a constant speed. As an object follows a circle with a given centre and radius, we need only one parameter to determine its position at any instant, e.g. the angle φ required to reach the point from some fixed direction (polar axis).

If we do **not** reset φ to zero with each complete rotation, we obtain a new quantity called **angular position**. By definition, rotation is positive anticlockwise. The angular position is measured in radians (rad). An **angular displacement** equals the difference between final and original angular position. One complete revolution agrees with the angular displacement 2π.

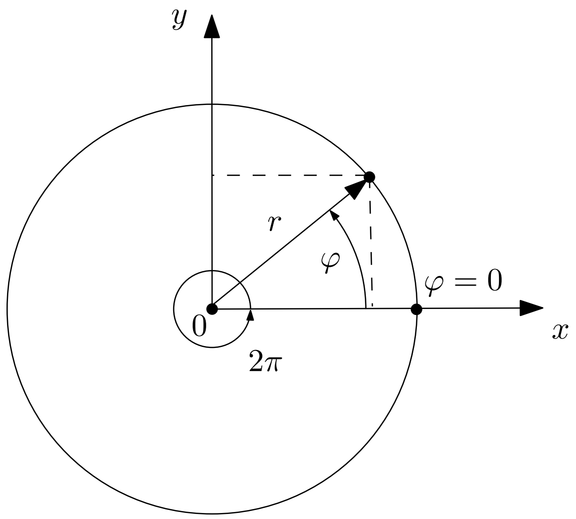


Figure 4.1: *Polar and Cartesian coordinate systems*

When describing a circular motion, it may be more convenient to use **polar coordinates** instead of the two Cartesian coordinates x and y. Look at figure 4.1. The polar coordinate system is a two-dimensional coordinate system in which each point on a plane is determined by an angle φ and a distance r from the centre. Henceforth, the angle will always be understood in radians rather than degrees. This choice is highly advantageous. For instance, it follows from the definition of radian (see table 1.2), that the length of an arc equals the central angle φ of the arc multiplied by its radius r, e.g. the circumference a circle equals $2\pi r$.

Polar coordinates may be converted to Cartesian coordinates by using the trigonometric functions sine and cosine:

$$\begin{aligned} x &= r\cos\varphi, \\ y &= r\sin\varphi, \end{aligned}$$

while Cartesian coordinates may be converted to polar coordinates using formulae

$$\begin{aligned} r &= \sqrt{x^2+y^2}, \\ \varphi &= \arctan\frac{y}{x}. \end{aligned}$$

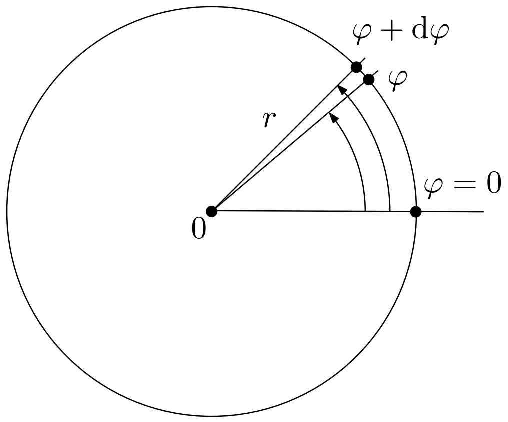


Figure 4.2: *Circular motion of a particle*

If the motion around the circle is uniform, the speed (magnitude of velocity) of the particle does not change. However, the velocity vector changes due to continuous change of its direction, as well as both of its components. As we can describe the position of a particle using just one angular variable – an angular position φ, we can also determine its speed using one variable which is an angular counterpart of velocity. We can define the average and the instantaneous angular velocities.

The average angular velocity of an object in the time interval Δt equals the the ratio of the total angular displacement $\Delta\varphi$ and the total time interval Δt over which the displacement takes place:

$$\omega = \frac{\Delta\varphi}{\Delta t}$$

Its unit is radian per second ($\mathrm{rad} \cdot \mathrm{s}^{-1}$).

The instantaneous angular velocity (or simply angular velocity) is the limit of the ratio of the angular displacement $\Delta\varphi$ and the time interval Δt over which the displacement takes place if the time interval Δt approaches zero, i.e. the derivative of φ with respect to t:

$$\omega = \lim_{\Delta t \to 0} \frac{\Delta\varphi}{\Delta t} = \frac{\mathrm{d}\varphi}{\mathrm{d}t}.$$

The following relation between speed and angular velocity can be found for a circular motion:

$$v = \frac{\mathrm{d}s}{\mathrm{d}t} = \frac{r\mathrm{d}\varphi}{\mathrm{d}t} = r\omega, \tag{4.1}$$

where the distance s is equal to the length of the arc.

In fact, the angular velocities (as well as angular acceleration, which we shall mention later) are vectors, although the definitions above refer onlyt to their magnitudes (and possibly two directions given by means of plus and minus signs). This approach is adequate if we study planar motion, as in rotation around a fixed axis. Even so, treating them as vectors may be useful. Angular velocity $\boldsymbol{\omega}$ is a vector pointing along the axis of rotation (normal to the circle). Its direction can be established using a right-hand rule: if you curl fingers of your right hand around the circle in the direction of rotation, your thumb (outstretched) points in the direction of the angular velocity vector.

So the angular velocity is a vector defining the axis as well as the speed of rotation.

Note the position vector (from the rotation centre) is always perpendicular to the angular velocity of a rotating point. If we consider the equation 4.1 and figure 4.3, we can realize that

$$\boldsymbol{v} = \boldsymbol{\omega} \times \boldsymbol{r}.$$

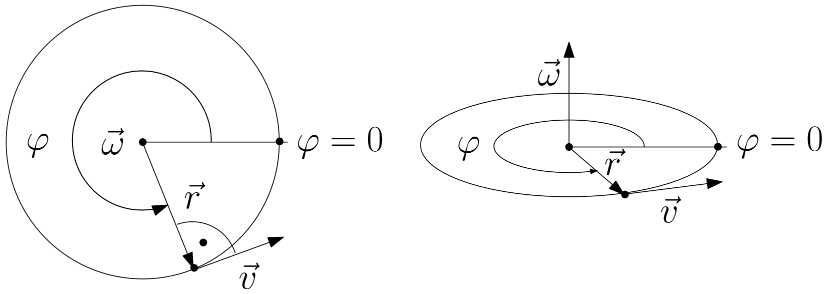


Figure 4.3: *Angular position φ, position vector $\boldsymbol{r}$, instantaneous velocity $\boldsymbol{v}$ and instantaneous angular velocity $\boldsymbol{\omega}$ (in the figure on the left pointing up to a reader) of a point rotating around the circle*

Sample problem 4.1 A merry-go-round rotates anticlockwise around a vertical axis z and it makes one revolution in $T = 10\,\mathrm{s}$. Express the coordinates of a horse, its representing point being at $t = 0\,\mathrm{s}$ at $x = 2\,\mathrm{m}$, $y = 0\,\mathrm{m}$, as functions of time. In addition, express the components of its velocity vector as functions of time.

Solution: The time T is called the **period** of revolution, it is the time an object needs to go around the closed path exactly once. Its reciprocal is the **frequency** f, it is the number of repeated events (e.g. revolutions) per unit of time. By complete analogy to linear motion, horse's angular position can be found as

$$\varphi(t) = \int \omega \,\mathrm{d}t = \omega t + C,$$

where the integration constant C can be determined considering the initial conditions. Looking at figure 4.1, it is obvious the values $r = 2\,\mathrm{m}$, $\varphi_0 = 0\,\mathrm{rad}$ will suit our purpose. So at $t = 0\,\mathrm{s}$ we have

$$\varphi(0) = \varphi_0 = 0 = \omega \cdot 0 + C \quad \Longrightarrow \quad C = 0,$$

therefore

$$\varphi(t) = \omega t.$$

The horse follows the circle with the constant angular velocity ω and it makes the angular distance 2π during one revolution, so it would follow that:

$$2\pi = \omega \cdot T$$

and

$$\begin{aligned} \omega &= \frac{2\pi}{T} \\ \varphi(t) &= \frac{2\pi t}{T}. \end{aligned}$$

Now we convert our result from polar coordinates φ and r to Cartesian coordinates x and y and get

$$\begin{aligned} x &= r\cos\varphi = r\cos\frac{2\pi t}{T} = 2\cos\frac{2\pi t}{10} = 2\cos\frac{\pi t}{5}, \\ y &= r\sin\varphi = r\sin\frac{2\pi t}{T} = 2\sin\frac{2\pi t}{10} = 2\sin\frac{\pi t}{5}. \end{aligned}$$

Remark: Note that the system of equations

$$\begin{aligned} x &= r\cos\omega t, \\ y &= r\sin\omega t \end{aligned}$$

represents the **parametric equations of a circle** in Cartesian coordinates, with time as a parameter. To find its single **implicit equation**, we must eliminate the parameter. This can be easily done by adding the squares of the two equations and, using the identity $\sin^2\alpha + \cos^2\alpha = 1$, we get the single equation

$$x^2 + y^2 = r^2.$$

The single equation no longer includes information about the position of our horse at some particular moment.

Now we need to find the velocity vector of the horse as a function of time. With the knowledge you already have in mathematics, there are two ways you can get the desired result. The first is to find the components of the velocity vector by means of differentiating its coordinates with respect to time

$$\begin{aligned} v_x &= \frac{\mathrm{d}x}{\mathrm{d}t} = \frac{\mathrm{d}}{\mathrm{d}t}\left(2\cos\frac{\pi t}{5}\right) = -\frac{2\pi}{5}\sin\frac{\pi t}{5}, \\ v_y &= \frac{\mathrm{d}y}{\mathrm{d}t} = \frac{\mathrm{d}}{\mathrm{d}t}\left(2\sin\frac{\pi t}{5}\right) = \frac{2\pi}{5}\cos\frac{\pi t}{5}. \end{aligned}$$

The second is to find the velocity vector as the vector product of the angular velocity and the position vector

$$\begin{aligned} \boldsymbol{\omega} &= \left(0,\ 0,\ \frac{\pi}{5}\right)\ \mathrm{rad}\cdot\mathrm{s}^{-1}, \\ \boldsymbol{r} &= \left(2\cos\frac{\pi t}{5},\ 2\sin\frac{\pi t}{5},\ 0\right)\ \mathrm{m}, \\ \boldsymbol{v} &= \boldsymbol{\omega}\times\boldsymbol{r}. \end{aligned}$$

The result is, of course, the same.

In the first chapter we stated that if an object moves along a curve (even at uniform speed), it undergoes acceleration, because its **velocity** vector changes. We mentioned a particle moving at a constant speed v along a circular path with the radius r possesses a centripetal acceleration. Now we can derive this acceleration.

Suppose a particle moves at constant speed around a circle. During a very short time $\mathrm{d}t$ it makes an angle $\mathrm{d}\varphi$ and its velocity vector changes from $\boldsymbol{v}$ to $\boldsymbol{v} + \mathrm{d}\boldsymbol{v}$ (see figure 4.4).

The length of the change in the velocity vector d$\boldsymbol{v}$ can be for a very small angular displacement $\mathrm{d}\varphi$ approximated by the arc length of radius v. It is the product of the radius and the angle (measured in radians), so $\mathrm{d}v \doteq v\mathrm{d}\varphi$. By analogy, the arc length of the moving particle is $r\mathrm{d}\varphi$. As the particle moves at a constant speed v during time $\mathrm{d}t$, its path can also be expressed as $v\mathrm{d}t$. So $\mathrm{d}\varphi = v\mathrm{d}t/r$ and the magnitude of the acceleration of a point moving uniformly along a circle is

$$a = \frac{\mathrm{d}v}{\mathrm{d}t} = \frac{v\mathrm{d}\varphi}{\mathrm{d}t} = \frac{v\frac{v\mathrm{d}t}{r}}{\mathrm{d}t} = \frac{v^2}{r}.$$

The direction of this acceleration is the same as the direction of d$\boldsymbol{v}$. It follows from the picture that it is perpendicular to the velocity vector and pointing to the centre of the circle (imagine the angle $\mathrm{d}\varphi$ really very small). So, it is **normal acceleration** (to the velocity vector) and it is called the **centripetal acceleration**.

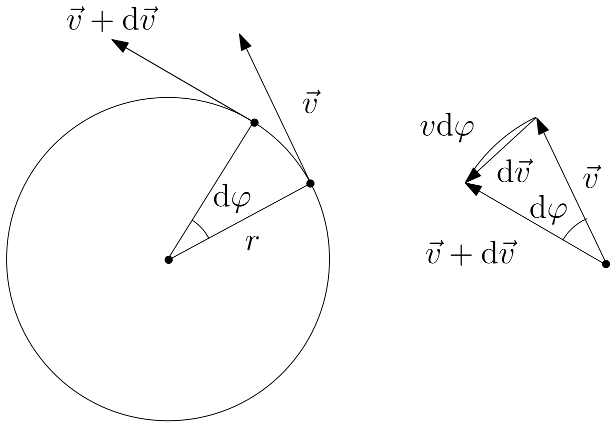


Figure 4.4: *The change of velocity vector at uniform circular motion*

! The **centripetal acceleration** of a point travelling around a circle with radius r at a constant speed v is a vector whose magnitude is

$$a_{\mathrm{c}} = \frac{v^2}{r}$$

and it is directed radially inward the circle.

To move along a circle some radial force is needed to create the appropriate centripetal acceleration. This force is called **centripetal force**. If there is a bullet rotating on a string, it is string tension (or its centripetal part). The Moon rotates around the Earth due to gravity. A car or a biker can turn thanks to friction. The centripetal force is proportional to the square of the velocity and inversely proportional to the radius. This is why on icy roads drivers unexpectedly skid during turns or when abruptly changing direction, friction being insufficient for the intention.

Sample problem 4.2 Velodromes for track cycling consist of two 180° circular bends connected by two straights. The turns are steeply banked to allow riders to keep their bikes relatively perpendicular to the surface while riding a curve at high speed. What would be the optimal tilt of a bend with radius $r = 30\,\mathrm{m}$ (considering the cyclist's centre

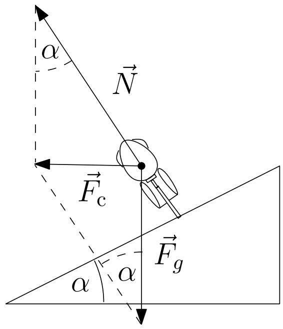


Figure 4.5: *A cyclist in a bend – the net force consisting of gravity* $\boldsymbol{F}_g$ *and the normal reaction force* $\boldsymbol{N}$ *from the track must be directed to the centre of the bend and its magnitude must correspond to the radius and the speed of the cyclist*

of mass) for the speed $v = 70\,\mathrm{km\cdot h^{-1}}$? What additional load will a cyclist experience at this point?
Solution: The optimal bank should match the natural lean of a bicycle moving through the curve. A cyclist should be able to go through the curve without employing the friction between the track and the wheels. The net force on the cyclist must be directed radially inward the circle his mass centre follows and its magnitude must create the appropriate centripetal acceleration. The net force consists of gravity $\boldsymbol{F}_g$ and the normal reaction force $\boldsymbol{N}$ from the track (see the figure 4.5). Therefore, the following must hold

$$F_\mathrm{c} = ma_\mathrm{c} = \frac{mv^2}{r}$$

and

$$F_\mathrm{c} = F_g \tan\alpha = mg\tan\alpha.$$

It follows

$$\alpha = \arctan\frac{v^2}{gr} = \arctan\frac{19.44^2}{9.81\cdot 30} = 52°6',$$

where the speed has been converted to basic units. In fact, the tracks are usually banked from approx. 30° to 45° (depending on the diameter of the curves), as the cyclists ride more slowly through curves before they reach maximum speed.

The force on the cyclist from his seat equals the normal force $\boldsymbol{N}$. The Pythagorean theorem implies that

$$N = \sqrt{F_g^2 + F_\mathrm{c}^2} = \sqrt{(mg)^2 + \left(\frac{mv^2}{r}\right)^2} = mg\sqrt{1 + \left(\frac{v^2}{gr}\right)^2}$$

Numerically, the ratio of the normal and gravitational forces equals

$$\sqrt{1 + \left(\frac{v^2}{gr}\right)^2} = 1.63.$$

As the forces felt by objects undergoing constant proper acceleration are indistinguishable from those in a gravitational field, the cyclist in the curve feels as if the acceleration of gravity has increased from $g = 9.81\,\mathrm{m\cdot s^{-2}}$ to $g' = 15.97\,\mathrm{m\cdot s^{-2}}$.

Now, let's come back to the theory of circular motion. Consider a particle travelling around a circle, but its motion is no longer uniform, its speed changes. In chapter 3 we studied motion along a straight line. We have seen that if an object accelerates, it possesses a positive acceleration (acceleration pointed in the direction of its motion), if an object decelerates, it possesses a negative acceleration (acceleration in the direction opposite to its motion). By analogy, if an object travels along a curve and its speed increases or decreases, it has a non-zero component of acceleration which is parallel to its velocity and it is called a **tangential acceleration**.

! The **tangential acceleration** at time t is defined as the derivative of speed v (the magnitude of velocity) with respect to time t

$$a = \frac{\mathrm{d}v}{\mathrm{d}t}.$$

In general, the acceleration has normal as well as tangential components (to the direction of motion).

We can also find this decomposition using calculus. The velocity vector $\boldsymbol{v}$ can be written as the product of the speed v and the unit tangent vector $\boldsymbol{\tau}$ and then differentiated with respect to time. We obtain the **vector of acceleration**

$$\boldsymbol{a} = \frac{\mathrm{d}\boldsymbol{v}}{\mathrm{d}t} = \frac{\mathrm{d}(v\boldsymbol{\tau})}{\mathrm{d}t} = \frac{\mathrm{d}v}{\mathrm{d}t}\boldsymbol{\tau} + v\frac{\mathrm{d}\boldsymbol{\tau}}{\mathrm{d}t} = a_{\mathrm{t}}\boldsymbol{\tau} + a_{\mathrm{n}}\boldsymbol{\nu},$$

where $\boldsymbol{\nu}$ is the unit vector normal to $\boldsymbol{\tau}$, as we will see shortly. The scalar product of an arbitrary unit vector with itself equals 1:

$$\boldsymbol{\tau} \cdot \boldsymbol{\tau} = 1,$$

so its differentiation yields 0. Using the rule for differentiation of products of functions we obtain

$$0 = \frac{\mathrm{d}(\boldsymbol{\tau} \cdot \boldsymbol{\tau})}{\mathrm{d}t} = \frac{\mathrm{d}\boldsymbol{\tau}}{\mathrm{d}t} \cdot \boldsymbol{\tau} + \boldsymbol{\tau} \cdot \frac{\mathrm{d}\boldsymbol{\tau}}{\mathrm{d}t} = 2\boldsymbol{\tau} \cdot \frac{\mathrm{d}\boldsymbol{\tau}}{\mathrm{d}t}.$$

As the scalar product of two non-zero vectors is only zero if they are perpendicular, the vectors $\boldsymbol{\tau}$ and $\mathrm{d}\boldsymbol{\tau}/\mathrm{d}t$ must be perpendicular. We can write

$$\frac{\mathrm{d}\boldsymbol{\tau}}{\mathrm{d}t} = \left|\frac{\mathrm{d}\boldsymbol{\tau}}{\mathrm{d}t}\right|\boldsymbol{\nu},$$

where $\boldsymbol{\nu}$ is the unit vector perpendicular to $\boldsymbol{\tau}$. The magnitude of $\mathrm{d}\boldsymbol{\tau}/\mathrm{d}t$ can be found in much the same way as the magnitude of centripetal acceleration.

During a very short time $\mathrm{d}t$ a point travelling around a circle makes an angle $\mathrm{d}\varphi$ and its velocity vector turns from the direction $\boldsymbol{\tau}$ to the direction $\boldsymbol{\tau} + \mathrm{d}\boldsymbol{\tau}$ (see figure 4.6). The length of the change of the unit tangent vector $\mathrm{d}\boldsymbol{\tau}$ can be (for a very small angular displacement $\mathrm{d}\varphi$) approximated by the arc length of radius τ, that is 1. So $\mathrm{d}\tau \doteq \tau\mathrm{d}\varphi = 1 \cdot \mathrm{d}\varphi = \mathrm{d}\varphi$. We have already shown that the length of the path of a rotating point equals $r\mathrm{d}\varphi = v\mathrm{d}t$. It follows $\mathrm{d}\varphi = v\mathrm{d}t/r$ and the magnitude of the change of the unit tangent vector will be

$$\frac{\mathrm{d}\tau}{\mathrm{d}t} = \frac{\mathrm{d}\varphi}{\mathrm{d}t} = \frac{\frac{v\mathrm{d}t}{r}}{\mathrm{d}t} = \frac{v}{r}.$$

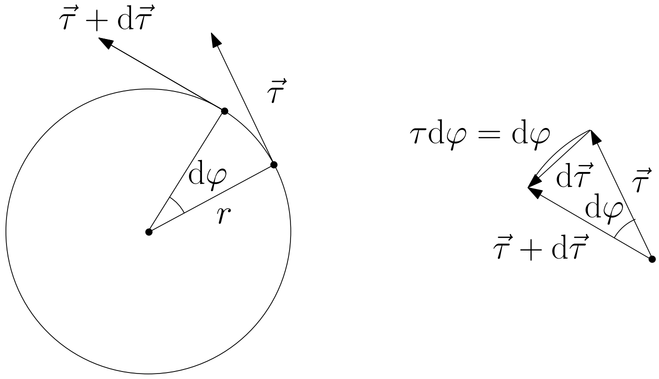


Figure 4.6: *During circular motion, the unit tangent vector* $\boldsymbol{\tau}$ *changes*

Finally, we get

$$a_{\mathrm{n}}\boldsymbol{\nu} = v\frac{\mathrm{d}\boldsymbol{\tau}}{\mathrm{d}t} = v\frac{v}{r}\boldsymbol{\nu} = \frac{v^2}{r}\boldsymbol{\nu},$$

as expected.

! **The instantaneous acceleration** of an object equals the rate of change of its velocity vector $\boldsymbol{v}$ in time:

$$\boldsymbol{a} = \frac{\mathrm{d}\boldsymbol{v}}{\mathrm{d}t} = a_{\mathrm{t}}\boldsymbol{\tau} + a_{\mathrm{n}}\boldsymbol{\nu},$$

where $\boldsymbol{\tau}$ is the unit tangent vector and $\boldsymbol{\nu}$ is the unit normal vector pointing radially inward the centre of the osculating circle (a circle whose part represents the best the momentary path of the object). Its components can be calculated as

$$\text{tangential acceleration :} \qquad a_{\mathrm{t}} = \frac{\mathrm{d}v}{\mathrm{d}t},$$
$$\text{normal acceleration :} \qquad a_{\mathrm{n}} = \frac{v^2}{r}.$$

If a particle accelerates around the circle, its angular velocity changes. It has an angular acceleration.

! The **angular acceleration** is the limit of the ratio of the change of angular velocity $\Delta\omega$ and the time interval Δt over which this change takes place if the time interval Δt approaches zero, i.e. the derivative of ω with respect to t:

$$\varepsilon = \lim_{\Delta t \to 0} \frac{\Delta\omega}{\Delta t} = \frac{\mathrm{d}\omega}{\mathrm{d}t}.$$

Its unit is radian per square second ($\mathrm{rad} \cdot \mathrm{s}^{-2}$).

In case of circular motion, the following useful formula holds

$$a_{\mathrm{t}} = \frac{\mathrm{d}v}{\mathrm{d}t} = \frac{\mathrm{d}(\omega r)}{\mathrm{d}t} = \frac{\mathrm{d}\omega}{\mathrm{d}t}r = \varepsilon r.$$

Sample problem 4.3 There is a glass on a table in an American limousine. The coefficient of static friction between the glass and the table is $\mu = 0.5$. The limousine starts to move along a horizontal curve with the radius $r = 200\,\mathrm{m}$ and it accelerates evenly at rate $2.5\,\mathrm{m} \cdot \mathrm{s}^{-2}$. How long will the glass remain at rest relative to the table?

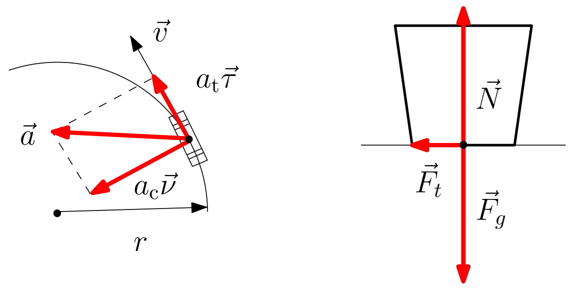


Figure 4.7: *On the left – the acceleration vector **a** of the car speeding up along a curve and its components. On the right – the sketch of forces acting on the glass (the origin of the gravity vector has been moved down to the bottom of the glass for clarity)*

Solution: The glass will stay at rest (with respect to the table) as long as the frictional force is sufficient to impart the same acceleration as the car has relative to the road. Its tangential component is known, $a_\mathrm{t} = 2.5\,\mathrm{m}\cdot\mathrm{s}^{-2}$. Its normal component gradually increases and it can be expressed as

$$a_\mathrm{n} = \frac{v^2}{r} = \frac{(a_\mathrm{t}t)^2}{r}.$$

The maximum frictional force is

$$F_\mathrm{t} = \mu N = \mu F_g = \mu m g,$$

so the maximum acceleration $a = \mu g$. It follows from the Pythagorean theorem that

$$\begin{aligned} a^2 &= a_\mathrm{t}^2 + a_\mathrm{n}^2 \\ (\mu g)^2 &= a_\mathrm{t}^2 + \left(\frac{(a_\mathrm{t}t)^2}{r}\right)^2 \end{aligned}$$

So the glass starts moving at

$$t = \sqrt[4]{\frac{r^2}{a_\mathrm{t}^2}\left(\left(\frac{\mu g}{a_\mathrm{t}^2}\right)^2 - 1\right)} = \sqrt[4]{\frac{200^2}{2.5^2}\left(\left(\frac{0.5\cdot 9.81}{2.5}\right)^2 - 1\right)}\ \mathrm{s} \doteq 11.6\ \mathrm{s}.$$

Problem 4.1 Determine the maximum uniform speed a car can go through a flat curved road with the radius $r = 30\,\mathrm{m}$, if the coefficient of static friction between the tyres and the icy road is $\mu = 0.3$.

Problem 4.2 A cable carousel at rest has the radius $r = 5\,\mathrm{m}$, the length of the cables on which the seats are fixed is $l = 3\,\mathrm{m}$. At the maximum speed of rotation the cables incline by $\alpha = 30°$. What is the frequency of rotation?

Problem 4.3 The rules for railway service allow the maximum uncompensated "tangential" acceleration of passengers $a = 1\,\mathrm{m}\cdot\mathrm{s}^{-2}$ (in this case "tangential" is understood as parallel to the floor, and is provided due to friction). To increase the maximum possible speed in turns the roadways of high-speed railways cant inward around the curves by up to 6°. Moreover, some modern trains are equipped with computer-controlled tilting mechanisms – the upper part of tilting trains can be tilted sideways. The Pendolino (the first tilting train to operate in the Czech Republic, see the figure 4.8), can tilt its upper part up to 8° [17]. What is the maximum speed it can go around a canted curve with a 300 m radius? How fast could an ordinary train take the curve?

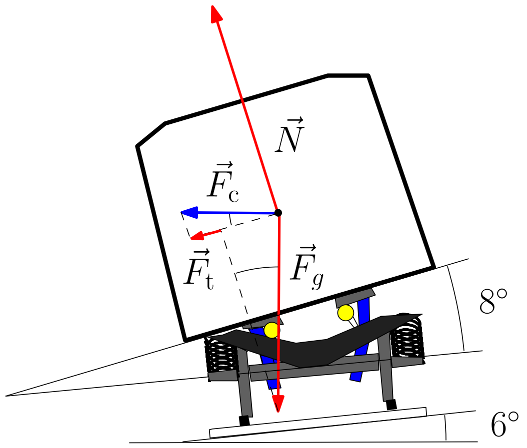


Figure 4.8: *A tilting train Pendolino in a curve and the forces acting on a passenger. The net force* $\boldsymbol{F}_\mathrm{c}$ *imparts the necessary centripetal acceleration. It consists of the gravity* $\boldsymbol{F}_g$*, the normal force from the seat* $\boldsymbol{N}$*, and, if the inclination is insufficient, the static friction* $\boldsymbol{F}_\mathrm{t}$*, restricted by rules (the origin of* $\boldsymbol{F}_\mathrm{t}$ *is moved for clarity)*

Problem 4.4 The Earth rotates around its axis, one revolution with respect to the distant stars takes approximately 23 hours, 56 minutes and 4 seconds (it is called the sidereal day and it is slightly shorter than the solar day because the Earth's orbital motion about the Sun). What is the speed of the points on the Earth's equator relative to the surrounding stars? The equatorial radius is about $R_\mathrm{E} \doteq 6378$ km. What is the average speed of the Earth relative to the Sun if their average distance is 1 AU $=1.496 \cdot 10^{11}$ m?

Problem 4.5 The Global Positioning System (GPS) utilises 24 satellites distributed equally among six circular orbits around the Earth. Their period of rotation is a half of the sidereal day, i.e. approx. 11 hours, 58 minutes and 2 seconds. What is their altitude? The equatorial radius of the Earth is about $R_\mathrm{E} \doteq 6375$ km, its mass $M_\mathrm{E} = 5.98 \cdot 10^{24}$ kg, and the gravitational constant $\kappa = 6.67 \cdot 10^{-11}\,\mathrm{N \cdot m^2 \cdot kg^{-2}}$. The satellites use no propulsion, except for stabilisation.

Problem 4.6 Geostationary satellites appear to be fixed over one spot above the Earth. Determine the altitude and the possible position of a geostationary satellite which has no propulsion.

4.2 Projectile Motion

Projectile motion refers to the motion of an object launched into the air. Neglecting drag, in the homogeneous gravitational field close to the Earth all projected objects have the same acceleration, called the free-fall acceleration $\boldsymbol{g}$. The gravitational force acting on them is $\boldsymbol{F}_g = m\boldsymbol{g}$, where m represents the object's mass.

Sample problem 4.4 As part of an action movie, a stuntman on a motorbike should run onto the torso of a bridge and make a long jump over a river. The maximum speed he can acquire on the damaged bridge is $v = 130\,\mathrm{km}\cdot\mathrm{h}^{-1}$, it is at $H = 10\,\mathrm{m}$ above the surrounding ground, the river is $d = 52\,\mathrm{m}$ wide. Can he make it?

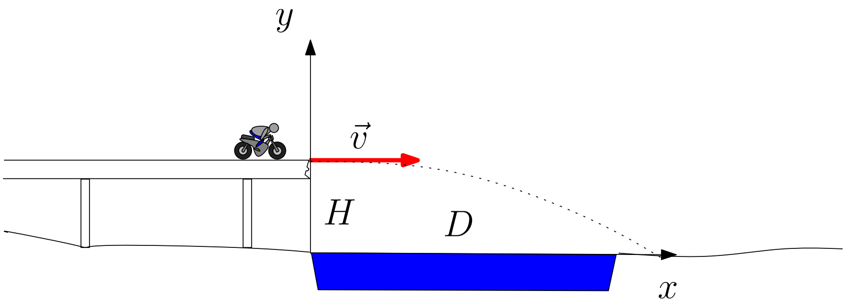


Figure 4.9: *A stuntman leaves the bridge with horizontal velocity* $\boldsymbol{v}$

Solution: The forces acting on the stuntman after leaving the bridge include gravity and air resistance. The air resistance will shorten the range of his jump in comparison to the path he would follow in a vacuum. To begin with, we examine whether he could jump over the river if there were no drag force. As there is no force in the horizontal x direction and the force in the vertical y direction is constant, the horizontal and vertical components of his motion can be analysed separately (see figure 4.9). The only force is gravity F_g, due to which the stuntman has the constant acceleration g against the y axis. Therefore his coordinates will evolve as follows

$$x = vt,$$
$$y = H - \frac{1}{2}gt^2,$$

where the initial position of the stuntman has been taken into account. To find the time of touchdown T we set $y = 0\,\mathrm{m}$ and we substitute the outcome to the second equation to obtain range D:

$$0 = H - \frac{1}{2}gT^2 \implies T = \sqrt{\frac{2H}{g}},$$
$$D = vT = v\sqrt{\frac{2H}{g}} = 35.1 \cdot \sqrt{\frac{2 \cdot 10}{9.81}}\,\mathrm{m} \doteq 50\,\mathrm{m},$$

all the variables have been filled in basic units. As it is less than the width of a river and in reality the jump will be even shorter, we see that the stuntman has no chance of accomplishing the task.

Now we shall seek a general solution for the motion of an object launched in a gravitational field with the initial velocity v_0. For simplicity, we will continue to disregard drag. We attempt to find the equation of the projectile's path. The coordinate system can be selected as in figure 4.10.

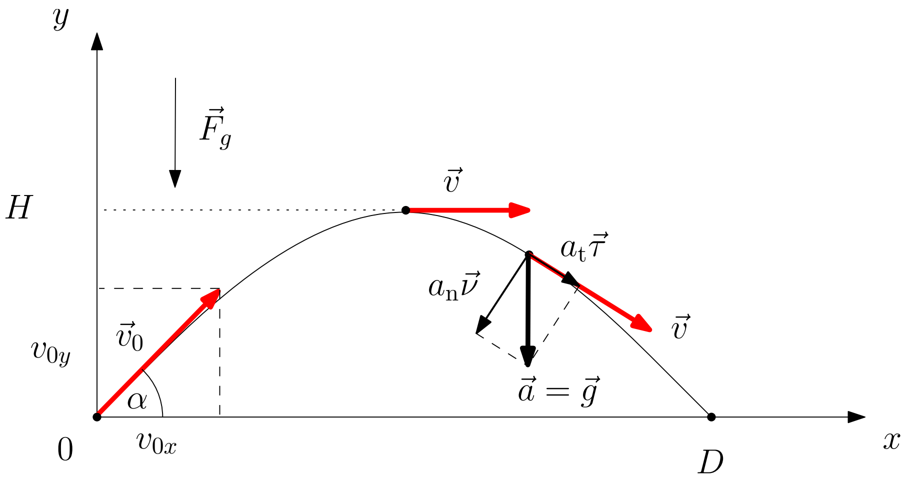


Figure 4.10: *An object is launched with initial speed* $\boldsymbol{v}_0$ *and it moves in a vertical plane with the free-fall acceleration* $\boldsymbol{g}$

The magnitude of the acceleration is $a = g$ and its components are

$$\begin{aligned} a_x &= 0, \\ a_y &= -g. \end{aligned}$$

The speeds in horizontal and vertical directions can be calculated separately

$$\begin{aligned} v_x(t) &= \int a_x \, \mathrm{d}t = \int 0 \mathrm{d}t = C_{1x}, \\ v_y(t) &= \int a_y \, \mathrm{d}t = \int (-g) \mathrm{d}t = -gt + C_{1y}, \end{aligned}$$

the unknown integration constants C_{1x} and C_{1y} must be determined from the initial conditions so that the velocity $\boldsymbol{v}(0)$ is equivalent to the given initial velocity $\boldsymbol{v}_0$. We have

$$\begin{aligned} v_x(0) &= C_{1x} = v_{0x}, \\ v_y(0) &= -g \cdot 0 + C_{1y} = v_{0y}, \end{aligned}$$

so the components of velocity are

$$\begin{aligned} v_x(t) &= v_{0x}, \\ v_y(t) &= v_{0y} - gt. \end{aligned}$$

Integrating the velocity components with respect to time we get the coordinates

$$\begin{aligned} x(t) &= \int v_x \, \mathrm{d}t = \int v_{0x} \mathrm{d}t = v_{0x} t + C_{2x}, \\ y(t) &= \int v_y \, \mathrm{d}t = \int (v_{0y} - gt) \mathrm{d}t = v_{0y} t - \frac{1}{2} g t^2 + C_{2y}, \end{aligned}$$

the unknown integration constants C_{2x} and C_{2y} can again be determined from the initial conditions, we now require that our object is at the origin of coordinates at $t = 0$

$$\begin{aligned} x(0) &= 0 = v_{0x} \cdot 0 + C_{2x}, \\ y(0) &= 0 = v_{0y} \cdot 0 - \frac{1}{2} g \cdot 0^2 + C_{2y}, \end{aligned}$$

so the coordinates of our flying object are

$$\begin{aligned} x(t) &= v_{0x}t, \\ y(t) &= v_{0y}t - \frac{1}{2}gt^2. \end{aligned}$$

So, what is the path of our projectile? We have its parametric equations in Cartesian coordinates, with time as a parameter. If your practice in mathematics is insufficient to recognise it, it may be easier for you if you find a single implicit equation. Solving the first equation for t and substituting into the second we get

$$\begin{aligned} t &= \frac{x}{v_{0x}}, \\ y &= v_{0y}\frac{x}{v_{0x}} - \frac{g}{2}\left(\frac{x}{v_{0x}}\right)^2 = Ax - Bx^2, \end{aligned}$$

which is the single equation of parabola. Finally, we can rearrange it to the standard form to see the vertex easily

$$\begin{aligned} (y - y_0) &= k\,(x - x_0)^2 \\ \left(y - \frac{v_{0y}^2}{2g}\right) &= -\frac{g}{2v_{0x}^2}\left(x - \frac{v_{0y}v_{0x}}{g}\right)^2. \end{aligned} \qquad (4.2)$$

Sample problem 4.5 A stone is launched from a horizontal plane with the initial speed $v_0 = 10\,\mathrm{m\cdot s^{-1}}$ at the angle of elevation $\alpha = 60°$. What is the total time of its flight T, the maximum height H and range D disregarding the influence of the air?
Solution: We can use the results of the previous case study. The components of velocity will be

$$\begin{aligned} v_x(t) &= v_{0x} = v_0\cos\alpha, \\ v_y(t) &= v_{0y} - gt = v_0\sin\alpha - gt. \end{aligned}$$

and the coordinates

$$\begin{aligned} x(t) &= v_{0x}t = v_0 t\cos\alpha, \\ y(t) &= v_{0y}t - \frac{1}{2}gt^2 = v_0 t\sin\alpha - \frac{1}{2}gt^2. \end{aligned}$$

To find the total time of flight T we set $y = 0$ – we search the intersection of the path with the horizontal x axis:

$$0 = y(t) = v_0\,T\sin\alpha - \frac{1}{2}gT^2 = T\left(v_0\sin\alpha - \frac{1}{2}gT\right).$$

The equation has two solutions, the first $T_1 = 0\,\mathrm{s}$ is the instant when the stone was launched, the second

$$T_2 = \frac{2v_0\sin\alpha}{g} = T$$

is the time of its touchdown. The range D equals the x coordinate at T:

$$D = x(T) = v_0\cos\alpha\,\frac{2v_0\sin\alpha}{g} = \frac{v_0^2\sin 2\alpha}{g}.$$

To find the maximum height of the stone we can employ elevation time T'. It can be found if we consider the zero-value of the y component of velocity at the highest point:

$$0 = v_y(T') = v_0 \sin\alpha - gT,$$

so

$$T' = \frac{v_0 \sin\alpha}{g}.$$

Note that the elevation time is half of the time of flight, which could be expected due to time symmetry of the flight.

The maximum height equals the y component at time T', so

$$H = y(T') = v_0 \sin\alpha \, \frac{v_0 \sin\alpha}{g} - \frac{g}{2}\left(\frac{v_0 \sin\alpha}{g}\right)^2 = \frac{(v_0 \sin\alpha)^2}{2g}.$$

We could also have used our knowledge of differential calculus to find the local extremum of the function $y(t)$. To find it, we must differentiate the function $y(t)$ with respect to time, set the derivative equal to zero and solve for time. Substituting the result back to the function $y(t)$ we get its maximum value. You may see that this procedure has much in common with our previous solution, as the derivative of y with respect to time is nothing other than the y component of velocity.

There is yet another way to find the maximum height H and range D of the stone. They can be read directly from the equation of parabola 4.2. The y component of the vertex corresponds to the maximum height H and the x component of the vertex is half of the range D.

$$\begin{aligned}\frac{D}{2} &= x_0 = \frac{v_{0y}v_{0x}}{g} = \frac{(v_0 \sin\alpha)(v_0 \cos\alpha)}{g} = \frac{v_0^2 \sin 2\alpha}{2g},\\ H &= y_0 = \frac{v_{0y}^2}{2g} = \frac{(v_0 \sin\alpha)^2}{2g}.\end{aligned}$$

Numerical values are: time of flight $T = 1.77\,\mathrm{s}$, the maximum height $H = 3.82\,\mathrm{m}$, and the range $D = 8.83\,\mathrm{m}$.

Sample problem 4.6 What is the smallest radius of curvature of the path of the stone from the previous sample problem?

Solution: The radius of curvature is connected with the acceleration by the relation

$$r = \frac{v^2}{a_\mathrm{n}}.$$

As the speed of the stone is minimal at the highest point and at the same time the normal acceleration is at its maximum there (all the acceleration $\boldsymbol{a} = \boldsymbol{g}$ is normal to the path at the top), the radius of curvature has its minimum there. It is

$$r = \frac{v_x^2}{g} = \frac{(v \cos\alpha)^2}{g},$$

and we get the result $r \doteq 2.55\,\mathrm{m}$.

In fact we cannot neglect the effects of the air in most cases. The disagreement between our results and the actual motion would be large especially for fast moving bodies of small densities, as the drag force is proportional to the square of speed

$$F_{\mathrm{d}} = \frac{1}{2} C S \varrho v^2.$$

The constant C depends on the shape of a body, S is the facing area and ϱ the density of the air. Then the calculation is much more complex and it yields the so-called **ballistic curve**. You can compare the two paths in figure 4.11.

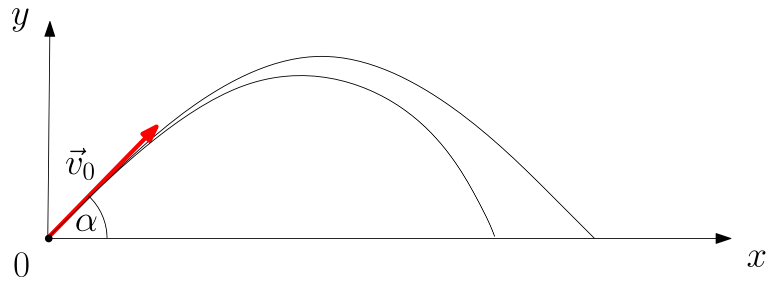


Figure 4.11: *The parabolic path of a projectile moving without air resistance and the ballistic curve the projectile will actually follow in the air*

Disregard the drag henceforth.

Problem 4.7 A ball is thrown directly upwards with an initial speed v_0. Determine the duration of its flight and its maximum height.

Problem 4.8 The body A starts falling down freely from an unknown height h_A. After $\triangle t = 0.5$ s the body B starts falling from the height $h_B = h_A - 4.9$ m. How long was the body A falling, if both bodies hit the ground simultaneously?

Problem 4.9 A particle is launched from a horizontal plane with the initial speed v_0 at the angle of elevation α. Determine the elevation angle that leads to the maximum horizontal range.

Problem 4.10 A particle was launched from a horizontal plane in such a way that the maximum height of its path was the same as its horizontal range. What was the angle of elevation α?

Problem 4.11 A stone was launched against the plane inclined by $\beta = 15°$ from horizontal. Its initial speed was $v_0 = 19.6 \ \mathrm{m \cdot s^{-1}}$ and the angle of elevation $\alpha = 45°$ (relative to horizontal). Determine the duration of the flight and the final distance of the stone from its origin.

Problem 4.12 (This problem is optional.) A firework explodes at $t = 0$ at the point $[0,\ 0,\ h]$ (z axis is pointed directly upward), so that its parts disperse themselves in all directions with approximately the same speed v. Determine the surface they will occupy later at some time t. Disregard the air resistance.

Chapter 5

Conservation Laws in Mechanics

What is cumulative action of force on a body?

What are further applications of the mathematics of ”diminutive changes”?

Why do the occupants of a lighter car have a much greater chance of serious injury or death in head-on collision?

Which quantities remain conserved in physical processes?

After finishing this chapter you should be able to

- define mechanical work, instantaneous power and average power and know their units,
- calculate the work done by a constant as well as a general variable force, and its power,
- express kinetic energy, gravitational potential energy and elastic potential energy, and solve simple problems using the law of conservation of mechanical energy,
- define (linear) momentum and angular momentum including their units,
- express how these quantities can be changed,
- decide when these quantities stay conserved and use the law of conservation of (linear) momentum and the law of conservation of angular momentum when appropriate.

5.1 Work, Power, and Energy

Sample problem 5.1 John Evans is a famous head-balancer and the holder of several Guinness World Records. In 1999 he balanced a Mini car weighing a total of 159.6 kg on his head for 33 seconds. What power was required for such an achievement? The heaviest snatch (weightlifting) of all time is 216.0 kg, lifted by Antonio Krastev of Bulgaria at the World Championships in Ostrava in 1987. What power has he demonstrated? What work has he done?

Figure 5.1: *John Evans balancing a Minicar on his head at one of his performances*

In our daily lives, the terms "work" and "power" are so broad, that they are accepted in senses far from their physical meaning. They can be used concerning physical activity as well as brainwork. In physics, these therms are much more strictly defined.

As we already know, every object tends to remain in a state of uniform motion unless some external force is applied to it. The change in motion is due to some non-zero net force. The final change in motion of a body depends on the magnitude and the direction of the acting net force as well as on the duration of its action. The cumulative effect of a force acting over some distance can also be described using a quantity called work. **Work** is a scalar quantity, expressing an effect of a force acting over some distance.

If the acting force $\boldsymbol{F}$ is constant, the **work** done by this force from position $\boldsymbol{r}_1$ to position $\boldsymbol{r}_2$ is (note a dot product between the force and the difference of position vectors)

$$W = \boldsymbol{F} \cdot (\boldsymbol{r}_2 - \boldsymbol{r}_1) = \boldsymbol{F} \cdot \Delta\boldsymbol{r} = F \Delta r \cos\alpha,$$

where α is the angle between $\boldsymbol{F}$ and $\Delta\boldsymbol{r}$.
The unit of work is the joule (J), $\mathrm{J} = \mathrm{kg} \cdot \mathrm{m}^2 \cdot \mathrm{s}^{-2}$.

Positive work means the force is acting in the direction of the motion (i.e. increasing its velocity). Negative work is when the force is acting opposite to the motion (i.e. reducing its velocity).

Well, how can we, in full generality, calculate the work W done from position $\boldsymbol{r}_1$ to position $\boldsymbol{r}_2$, if the acting force is not constant? In much the same way as when calculating the total displacement of a point moving with variable speed during a time interval. We first divide our position interval into n small subintervals $(\boldsymbol{r}_i, \boldsymbol{r}_i + \Delta\boldsymbol{r}_i)$, within which we assume the force to be approximately constant, $\boldsymbol{F}_i$. Then $W_i = \boldsymbol{F}_i \cdot \Delta\boldsymbol{r}_i$ and the total work

$$W \doteq \sum_{i=1}^{n} W_i = \sum_{i=1}^{n} \boldsymbol{F}_i \cdot \Delta\boldsymbol{r}_i.$$

To get more accurate result, we refine our partition. By making better and better approximations (shorter and shorter position subintervals), we obtain in the limit infinite number of infinitesimal position intervals $\Delta\boldsymbol{r}_i \rightarrow \mathrm{d}\boldsymbol{r}$, and our summation turns into integration.

If the acting force $\boldsymbol{F}$ is not constant, the work done by this force over the infinitesimal position change $\mathrm{d}\boldsymbol{r}$ is

$$\mathrm{d}W = \boldsymbol{F} \cdot \mathrm{d}\boldsymbol{r}$$

and to calculate the total work done from position $\boldsymbol{r}_1$ to position $\boldsymbol{r}_2$, we must integrate the work over the motion path

$$W = \int_{\boldsymbol{r}_1}^{\boldsymbol{r}_2} \boldsymbol{F}(\boldsymbol{r}) \cdot \mathrm{d}\boldsymbol{r} \,.$$

In general, the calculation of work done by variable force may be difficult.

The quantity describing the rate at which work is done by a force (e.g. when you are lifting a weight) is called power.

Average power is the ratio of work W and the time interval Δt, over which this work was done:

$$\overline{P} = \frac{W}{\Delta t} \,.$$

The unit of power is the watt (W), $\mathrm{W} = \mathrm{kg} \cdot \mathrm{m}^2 \cdot \mathrm{s}^{-3}$.

Average power is not a suitable quantity when you are interested in the power at a particular moment. The shorter the time interval Δt we take into account, the more accurate is the information about power at that moment provided by the ratio $\Delta W/\Delta t$. In a limit $\Delta t \to 0$ we obtain instantaneous power, which is derivative of work with respect to time.

Instantaneous power (or simply power) is the derivative of work with respect to time:

$$P = \frac{\mathrm{d}W}{\mathrm{d}t} \,.$$

In case of translation motion one more formula can be derived for instantaneous power, as follows:

$$P = \frac{\boldsymbol{F} \cdot \mathrm{d}\boldsymbol{r}}{\mathrm{d}t} = \boldsymbol{F} \cdot \boldsymbol{v}.$$

Solution to sample problem 5.1: It follows from the definition of work that no work was done by John Evans balancing a car on his head and the power also was zero. The work would be zero even if he had walked round the stage (disregarding the drag force of the air, which would have been very small due to very low speed). The force exerted by John Evans would have been perpendicular to the displacement and their scalar product equal to zero.

The power of Antonio Krastev can be estimated just approximately. Suppose he lifted 216.0 kg mass weight for 2 meters and it took roughly 1.5 second. The weight of 216 kg mass is about 2100 N (this approximation will suffice as we simply estimate the height and time). If we suppose the motion was vertically upwards and the lifting force was constant (although the force would surely have varied during the motion, the total work is given by its average). The total work done by Krastev was roughly

$$W = \boldsymbol{F} \cdot \Delta\boldsymbol{r} = F\Delta r = 2100\,\mathrm{N} \cdot 2\,\mathrm{m} = 4200\,\mathrm{J}$$

and the average power

$$\overline{P} = \frac{W}{t} = \frac{4200\,\mathrm{J}}{1.5\,\mathrm{s}} = 2800\,\mathrm{W}.$$

We cannot determine the instantaneous power without more detailed information about the course of the snatch.

Sample problem 5.2 A boy pulls a box of weight $G = 500$ N across a horizontal floor for the distance $s = 6$ m using a rope that is inclined at an angle $\alpha = 30°$ from horizontal. The coefficient of kinetic friction between the box and the floor is $\mu_\mathrm{k} = 0.3$. What is the total work the boy does on the box?

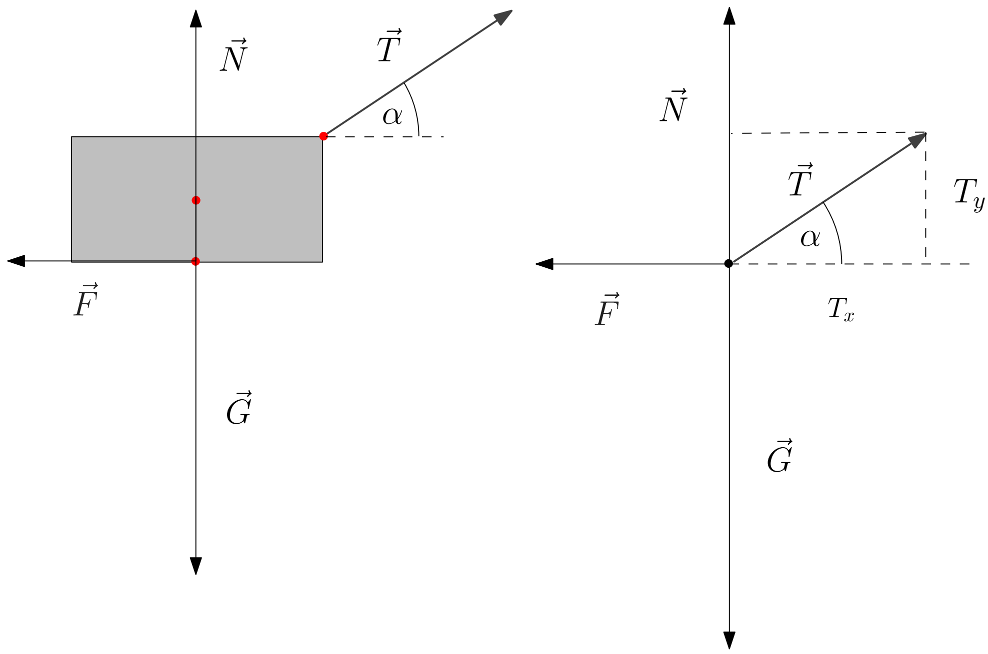


Figure 5.2: *A box pulled via an inclined rope*

Solution: Suppose the boy moves the box with a constant speed. In fact, he must set the box in motion at first and stop it in the end, but the total work necessary to shift the box over the given distance will be the same. If the acting force $\boldsymbol{T}$ is assumed to be constant, the work done by this force over the distance s is

$$W = T \cdot s \cdot \cos\alpha,$$

where α is the angle between the force and the displacement. Now we must determine the magnitude of the tension force $\boldsymbol{T}$. Let's draw a box and all the forces applied to it. They include the tension force $\boldsymbol{T}$, gravitational force $\boldsymbol{G}$, normal reacting force from the floor $\boldsymbol{N}$ and frictional force $\boldsymbol{F}$. If we suppose the box moves with a steady motion, the net force must be zero. We select a coordinate system so that x axis is horizontal and y axis vertical. It follows

$$\begin{aligned} F &= T_x = T\cos\alpha, \\ N &= G - T_y = G - T\sin\alpha, \\ F &= \mu \cdot N. \end{aligned}$$

Solving this system of equations we get

$$T = \frac{\mu G}{\cos\alpha + \mu\sin\alpha}$$

and the work

$$W = \frac{\mu G s}{1 + \mu \tan \alpha}.$$

Finally we fill in the given values and obtain

$$W = \frac{0.3 \cdot 500 \cdot 6}{1 + 0.3 \tan 30^\circ}\,\mathrm{J} = 767\,\mathrm{J}.$$

Remark: Note that the same but **negative** mechanical work is done on box by the floor (via frictional force) at the same time. Thus the total mechanical work done on the shifted box is zero and it is at rest at the beginning as well as at the end of the action. Due to friction, the work done by the boy simply produces an increase of the internal energy of the box and the floor (we shall not discuss this kind of energy in detail here). Frictional force is one of non-conservative forces, which you will study in more detail in the Physics I course.

Now, suppose a constant force is exerted on a body on a frictionless horizontal surface.

Sample problem 5.3 What is the work done by a constant force which sets a body of mass m, initially at rest, into motion with speed v?

Solution: At the beginning the body is at rest. Due to Newton's second law, the constant force F accelerates the body with the constant acceleration $\boldsymbol{a} = \boldsymbol{F}/m$. If the original velocity is zero, the velocity v attained during a time interval is the integration of acceleration a over that time interval (to distinguish the integration variable from the integration limit we name the first t' and the second t)

$$v = \int_0^t a \mathrm{d}t' = a \int_0^t \mathrm{d}t' = a\,[t']_0^t = at.$$

The distance then grows as

$$s = \int_0^t v(t')\mathrm{d}t' = \int_0^t at'\mathrm{d}t' = a \int_0^t t'\mathrm{d}t' = a\left[\frac{t'^2}{2}\right]_0^t = \frac{1}{2}at^2.$$

As the force is constant, total work can be calculated as

$$W = \boldsymbol{F} \cdot \Delta\boldsymbol{r} = Fs = (ma) \cdot \frac{1}{2}at^2 = \frac{1}{2}m(at)^2 = \frac{1}{2}mv^2.$$

Remark: In this case the work done on the body by the force has accelerated the body to motion with some speed. A scalar quantity called kinetic energy is associated with such a state of motion.

! **Kinetic energy (of translation)** of a body equals the work necessary to accelerate the body, initially at rest, to motion with a given speed. A body of mass m moving with speed v has kinetic energy of translation

$$E_k = \frac{1}{2}mv^2.$$

The unit of energy is the same as the unit of work, i.e. J (joule).

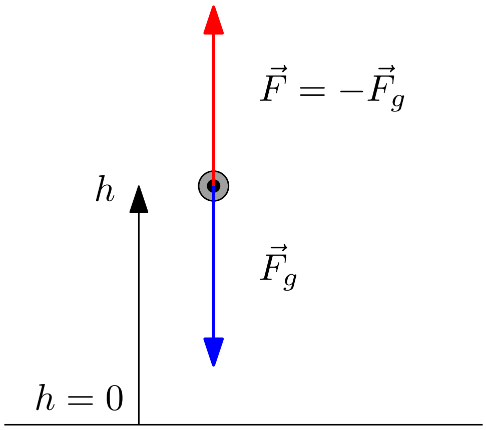


Figure 5.3: *In lifting a body in homogeneous field of gravity we have to apply on it a force opposite to gravitational force*

Sample problem 5.4 Suppose a homogeneous field of gravity and a particle-like body of mass m to be lifted to a certain height h. What work must be done on a body by the lifting force?
Solution: As the gravitational force $\boldsymbol{F}_g$ points downwards and its magnitude is $F_g = mg$, to lift a body evenly in gravitational field we must apply the force $F = mg$ pointing upwards, in the direction of motion. The work done by this force is

$$W = F\,h = m\,g\,h.$$

Since gravitational force is perpendicular to the Earth's surface, no work is generated in horizontal directions. Therefore, if we move an object up in such a way that the displacement $\boldsymbol{d}$ makes an angle α with the vertical, we still obtain the work $W = m\,g\,h = m\,g\,d\cos\alpha$, where h is the height difference between the final and the initial position.

Remark:

In this example the work was necessary to change the position of a body in some field, namely the homogeneous field of gravity. It induced the growth of the potential energy of the body. Note that the same but **negative** work was done on the body by the gravitational force. Therefore the speed of the body has not changed and there is no change in kinetic energy of the body. If we release the body from on high, it will fall down and attain speed (due to work of gravitational force). Its potential energy in the gravitational field changes into its kinetic energy.

! **Gravitational potential energy** of a body is equal to work necessary to lift the body to the height of h in the gravitational field of the Earth. A body of mass m at height h in the homogeneous gravitational field has the potential energy

$$E_p = mgh,$$

where g is the free-fall acceleration.

A conservative force means that the work done by the force is independent of the path. In fields of conservative forces potential energy can be defined. The change in potential energy can be set equal to work done by the external force (acting against the field force) to move a body from one point to another. As only changes in potential energy play a role in physics (e.g. result in a change of kinetic energy), to determine the potential energy unambiguously, we have to select a reference point or level where we set the potential energy equal to zero. It can be the surface of the Earth, the floor or any other level suitable for the given situation.

! **Potential energy** is energy stored within a physical system. The potential energy of a body (due to a field generated by another body) equals the work necessary to move the body from the zero-energy reference point to its current position.

It follows from Newton's universal law of gravitation that a gravitational field can be supposed homogeneous only within a small range of distances (e.g. from the surface of the Earth). For longer distances we must take into account that the strength of a gravitational field varies with location.

Sample problem 5.5 What work must be done on a body of mass m to move it from $r \geq R_Z$ from the Earth's centre to infinity?

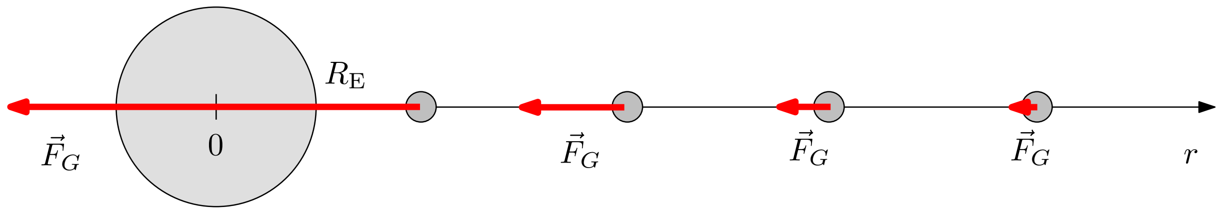


Figure 5.4: *In lifting a body in a general field of gravity we have to apply on it a variable force opposite to the gravitational force $\boldsymbol{F}_G$ (in the picture)*

Solution: In lifting a body steadily in a general field of gravity we have to apply on it a variable force opposite to Newton's gravitational force

$$F_G = \kappa \frac{mM}{r^2},$$

in the direction of motion. The work done by this force will be

$$W = \int_r^\infty F_G \, \mathrm{d}r',$$

where the limits r and ∞ indicate the initial and the final distances from the Earth's centre and the integration variable is denoted r'. Therefore

$$W = \int_r^\infty \kappa \frac{mM}{r'^2} \, \mathrm{d}r' = \kappa mM \int_r^\infty \frac{1}{r'^2} \, \mathrm{d}r' = \kappa mM \left[-\frac{1}{r'} \right]_r^\infty = \kappa \frac{mM}{r}.$$

Newton's gravitational force decreases as the inverse square of the distance from the gravitating body and it is zero in infinity. It is therefore practical to select the zero reference point there. The gravitational potential energy elsewhere will then be negative and equal $E_p = -\kappa mM/r$.

The gravitational potential energy of an object at a distance r from the centre of the Earth relative to its potential energy at an infinite distance away from the Earth is equal to the work necessary to move it from infinity to the distance r from the Earth. It is given by the expression

$$E_p = -\kappa \frac{mM}{r},$$

where κ is the gravitational constant, m is the mass of the object and M is the mass of the Earth.

In much the same way the potential energies due to other objects can be expressed, e.g. the Sun, the Moon, etc.

Finally, we shall introduce yet another type of potential energy – elastic potential energy. Like in the case of gravitational potential energy, elastic potential energy of a spring equals the work necessary to stretch it from its relaxed length (where its elastic potential energy is zero) to the desired length.

Sample problem 5.6 If Hooke's law holds (it holds for small enough elongations) the applied force is proportional to elongation of a spring. The proportionality constant is called a spring constant. What work must be done to lengthen a spring of spring constant k for the length d from its relaxed state?

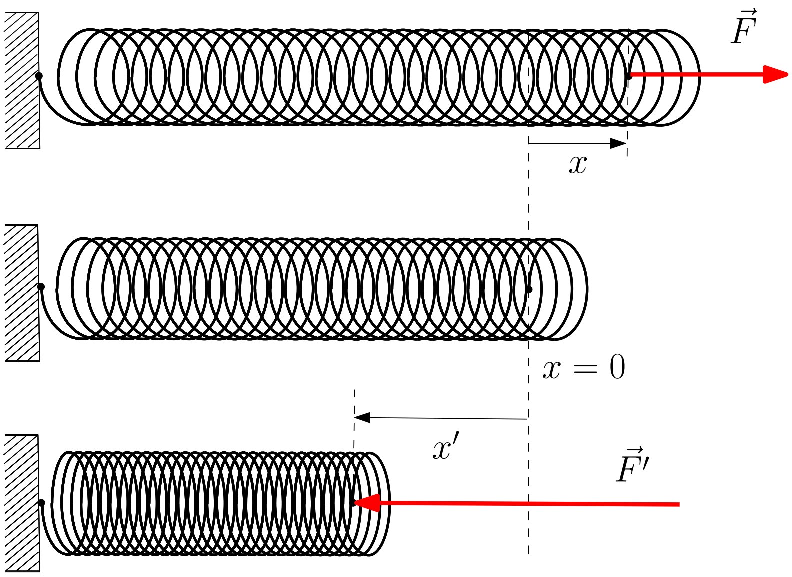


Figure 5.5: *A spring stretched, in relaxed state, and compressed. Elongation is proportional to the applied force*

Solution: If we want to stretch a spring, we apply on it the force $F = kx$ in the direction of elongation, where x is the instantaneous elongation of the spring. As the force is not constant, the work has to be calculated using integration

$$W = \int_0^d F(x)\,\mathrm{d}x = \int_0^d k\,x\,\mathrm{d}x = k\int_0^d x\,\mathrm{d}x = k\left[\frac{x^2}{2}\right]_0^d = \frac{1}{2}kd^2.$$

The same result can be obtained considering the average tension force.

If the spring is to be stretched or compressed a small distance d from its relaxed length, a force on the spring according to Hooke's law is

$$F = kd,$$

where k a spring constant. The work necessary to stretch the spring to this state equals its elastic potential energy

$$E_p = \frac{1}{2}kd^2.$$

We have introduced several processes where the work done on a body changed its energy. Let's now consider a system of bodies which do not interact with any external bodies, no work is done on the system by external forces. This system is called an **isolated system**. As there can be no energy transfers to or from an isolated system, the total energy of such a system (including kinetic, potential and internal energies) cannot change.

The law of conservation of energy
The total energy of an isolated system cannot change

$$\Sigma E = \text{const.}$$

If there are only conservative forces between the parts of an isolated system (there are no frictional, drag or other dissipative forces) there is no energy transfer from **mechanical energy**, including kinetic energy and various types of potential energy, to other forms of energy.

The law of conservation of mechanical energy
Total mechanical energy of an isolated system with only conservative forces within the system cannot change.

$$\Sigma E_k + \Sigma E_p = \text{const.}$$

The law of conservation of energy can be very useful when solving various physical problems and may simplify the solution great deal, as in the following sample problem.

Sample problem 5.7 What is the minimum speed a cyclist must have at the entry of a loop track with a radius $R = 5$ m, to get safely through? In the loop the cyclist does not pedal. The centre of gravity of the cyclist and his bike is 1.2 m above the ground. Suppose that friction, drag and inertia of wheels can be disregarded.
Solution: The cyclist's centre of mass goes along a circle with a radius $R_\mathrm{C} = (5-1.2)$ m $=$ 3.8 m. Some minimal non-zero speed is needed to keep him from falling off the track at each point. The topmost point, where the speed is lowest and the gravity is directed to the centre of loop, is critical. We already know the relation between the centripetal force, mass, radius and the speed of an object moving along a curve

$$F_\mathrm{c} = \frac{mv_2^2}{R_\mathrm{C}}.$$

The centripetal force acting on a cyclist at the top of the loop is at least equal to the gravitational force $F_\mathrm{c} \geq F_g = mg$. Therefore his minimum speed there must be

$$v_{2\min}^2 = gR_\mathrm{C},$$

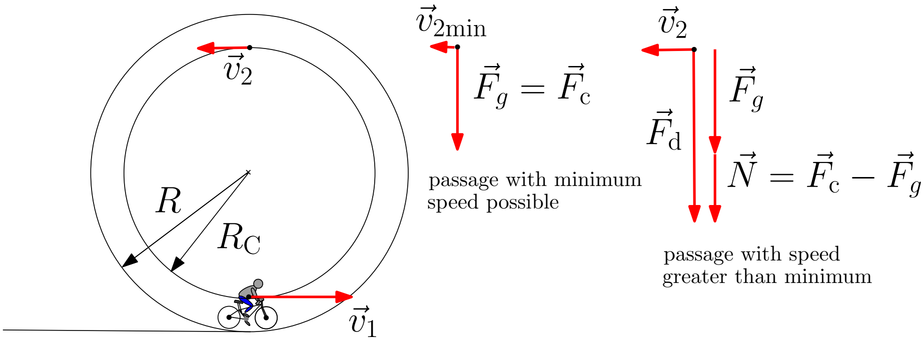


Figure 5.6: *A cyclist riding through a loop track and forces acting on him at the topmost (critical) point of the track*

otherwise his radius will be less then R_C and he will fall off the track. If his speed at the top is greater, it causes no problem, the remaining part of the requisite centripetal force being provided by the normal force $\vec{N}$ from the loop (see figure 5.6).

Now, we know the necessary speed at the top of the track, but how can we determine the initial speed of the cyslist at the bottom? It would be rather complicated to calculate how he slows down along his trajectory. Fortunately, we don't need to know the entire history of his motion. The relation between his speeds at the top and at the bottom of the track can be easily determined using the law of conservation of mechanical energy.

We select a zero-level of potential energy at the centre of mass of a cyclist when he is entering the loop. At the top his centre of gravity is $2R_C$ above this level. His total mechanical energy at these points includes kinetic and potential energies

$$\begin{aligned} E_{k1} + E_{p1} &= E_{k2} + E_{p2}, \\ \frac{1}{2}mv_1^2 + 0 &= \frac{1}{2}mv_2^2 + 2mgR_C, \end{aligned}$$

so the relation between the two speeds is

$$v_{1\min}^2 = v_{2\min}^2 + 4gR_C = gR_C + 4gR_C = 5gR_C.$$

We get the result

$$v_{1\min} = \sqrt{5gR_C} \doteq 13,7\,\mathrm{m \cdot s^{-1}} \doteq 49\,\mathrm{km \cdot h^{-1}}.$$

Problem 5.1 The longest escalator in the Czech republic is at the underground station Peace Square, it is 87.2 m long and inclined at 30° from horizontal. What is the work done by the escalator as a person weighing 70 kg ascends?

Problem 5.2 A person drives a car with speed $v = 90\,\mathrm{km \cdot h^{-1}}$. Suddenly, he spots an obstacle on the road and brakes heavily. As the road is wet, the car skids, the coefficient of kinetic friction between the tyres and the road being only $\mu = 0.55$. What is the braking distance? Solve considering the kinetic energy and the negative work done on the car by the frictional force.

Problem 5.3 A building lift carries up to the third floor ($h = 10\,\mathrm{m}$) a barrel filled with water, it takes 15 s. There is a small leakage at the bottom of the barrel, so its mass decreases as $m = 4(50 - t)^2\,\mathrm{kg, s}$. What is the work done by the lift?

Problem 5.4 Find the energy necessary to fix a GPS satellite in its orbit. The mass of the satellite is 775 kg, it circles the globe at about 20200 km above the Earth's surface with the speed $v = 3875\,\mathrm{m \cdot s^{-1}}$. The radius of the Earth is $R_\mathrm{E} \doteq 6375$ km, gravitational constant $\kappa = 6.67 \cdot 10^{-11}\,\mathrm{N \cdot m^2 \cdot kg^{-2}}$, mass of the Earth $M_\mathrm{E} = 5.98 \cdot 10^{24}$ kg, suppose the globe is spherical (in fact, the distance between its equatorial and polar diameters is about 43 km).

Problem 5.5 What is the instantaneous power at $t = 3\,\mathrm{s}$ if we stretch a spring from its relaxed state with the constant speed $v = 0.05\,\mathrm{m \cdot s^{-1}}$? The spring constant is $k = 5\,\mathrm{N \cdot cm^{-1}}$. What is the average power from $t_0 = 0\,\mathrm{s}$ to $t = 3\,\mathrm{s}$?

Problem 5.6 What is the maximum excess load of a jumper on a giant swing from sample problem 2.7?

Problem 5.7 A particle-like body of mass m slides down a frictionless inclined plane, which turns into a loop with a radius R (similar to the sample problem 5.7). What must be its initial height h to be able to remain on the track around the whole loop, if its initial speed is v_0?

Problem 5.8 A Renault Scenic 2.0 16V RX4 Salomon weighing 1000 kg has power $102\,\mathrm{kW} = 102\,\mathrm{kJ \cdot s^{-1}}$. What is the maximum slope where it can maintain the speed $72\ \mathrm{km \cdot h^{-1}}$? Disregard the drag.

Problem 5.9 A package chute has been installed at a distribution centre to move packages from the second floor to the first floor. It consists of an inclined plane and a horizontal stop-way. The mass of the packages is $m = 25$ kg, the height of the chute $h = 3$ m, a sloped part is inclined at $\alpha = 30°$ from horizontal, coefficient of dynamic friction between the packages and the chute is $\mu_D = 0.5$. What is the distance a package travales on the horizontal part of the chute until it stops? Assume a smooth transition from the sloped to the horizontal section of the chute, disregard the size of the package.

Problem 5.10 A bungee-jumper examines a rope of relaxed length $l_0 = 20$ m before her jump. If she is hanging steadily on it, the rope becomes longer for $\Delta l = 5$ m. Assuming Hooke's law holds (i.e. the elongation is proportional to the tension force) decide if she can jump safely on this rope from the bridge which is $h = 35$ m high (she starts from the same point at which the rope is fixed). What would be the maximum elongation of the rope under these conditions?

5.2 Momentum and Angular Momentum

Police statistics show that if there is a head-on collision of a heavy and a light car, the occupants of the lighter car are much more likely to be seriously or even fatally injured. What takes place in such a collision?

Newton's second law is concerned with the relation between a force $\boldsymbol{F}$, a mass m of an object and its acceleration $\boldsymbol{a}$ (the time rate of change of its velocity)

$$\boldsymbol{F} = m\,\boldsymbol{a} = m\frac{\mathrm{d}\boldsymbol{v}}{\mathrm{d}t}. \tag{5.1}$$

Figure 5.7: *The result of a car crash. The delivery van has stopped almost on the spot in its original track, but the passenger car, which, before the crash had been travelling along the solid line, has ricocheted, and it has been turned around by approx.* 90°. *Luckily, nobody was seriously injured.*

The same law can be expressed using a quantity called (linear) momentum.

! **The (linear) momentum of a particle** is a vector quantity that is defined as the product of its mass m and its velocity $\boldsymbol{v}$

$$\boldsymbol{p} = m\boldsymbol{v}.$$

The unit for linear momentum is $\mathrm{kg} \cdot \mathrm{m} \cdot \mathrm{s}^{-1}$.

So Newton's second law can be expressed as follows.

! The time rate of change of the momentum of a particle is equal to the net force acting an the particle

$$\frac{\mathrm{d}\boldsymbol{p}}{\mathrm{d}t} = \boldsymbol{F}.$$

This formulation is even more general then the relation 5.1, as it includes the possibility of mass change. This is important for objects moving with speed close to the speed of light, as well as for the objects propelled by rocket engines.

Sample problem 5.8 Determine the change of the linear momentum of an 80 kg occupant of a car, which frontally crashes to the wall with speed $v = 50\,\mathrm{km} \cdot \mathrm{h}^{-1}$. Estimate the duration of braking of the person secured with a seat belt if you know that the front of the car has been deformed by $L = 65\,\mathrm{cm}$ and the seat belt has given a passenger $l = 15\,\mathrm{cm}$ due to suspension. What was the average force acting on the passenger during the crash?
Solution: In the end of the crash the passenger has zero momentum, therefore the momentum change equals its value before crash

$$\Delta p = m \cdot v = 80 \cdot 13.9\,\mathrm{kg} \cdot \mathrm{m} \cdot \mathrm{s}^{-1} = 1112\,\mathrm{kg} \cdot \mathrm{m} \cdot \mathrm{s}^{-1},$$

where we have converted speed to SI units. To estimate the braking force on the passanger we can consider his original kinetic energy and the work of the braking force F, which is negative and causes him to stop. Only the mean value can be estimated, using the relation

$$\frac{1}{2}mv^2 = F \cdot s,$$

where the braking distance $s = L + l$. It follows

$$F = \frac{mv^2}{2s} = 9.66\,\text{kN}$$

and braking time if we suppose the braking force to be constant

$$\Delta t = \frac{\Delta p}{F} = 0.12\,\text{s}.$$

We realise that in a frontal crash, deformation of the frontal part of the car (except the compartment) is desirable, as it absorbs part of the energy, protracts the braking distance and time of the car's occupants and thus restrains the braking force on them. Wearing a seat belt is crucial to reduce the risk of serious or fatal injury. The seat belt limits the forward movement of the body, thus avoiding contact with the hard interior of the car and limits the forces placed to the body. There are also systems including load limiters that allow the belt to unreel in a controlled way so that the pressure on the occupant's chest area is further reduced [19].

Sample problem 5.9 A car weighing $m = 1200\,\text{kg}$ turns right with a constant speed $v = 54\,\text{km}\cdot\text{h}^{-1}$. What is the magnitude of the change in its linear momentum?

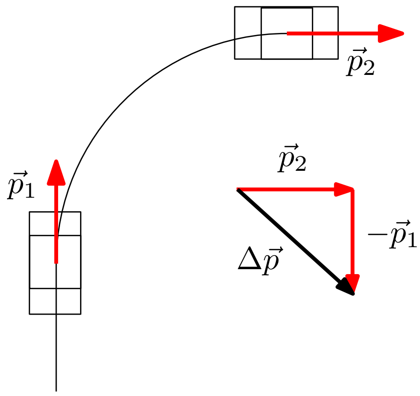


Figure 5.8: *The change of the linear momentum of a car in a right-hand right-angle turn*

Solution: The magnitude of the linear momentum of the car does not change, however, the vector of linear momentum changes due to the change in its direction (see figure 5.8). The change of the linear momentum is

$$\Delta \boldsymbol{p} = \boldsymbol{p}_2 - \boldsymbol{p}_1.$$

We can subtract the two vectors geometrically by adding the vector opposite to $\boldsymbol{p}_1$ to the vector $\boldsymbol{p}_2$. The magnitude of change of the linear momentum is the hypotenuse of the equilateral right-angled triangle with initial and final momenta as its legs, it follows

$$\Delta p = \sqrt{p^2 + p^2} = \sqrt{2}p = \sqrt{2}mv = \sqrt{2}\cdot 1200\,\text{kg}\cdot 15\,\text{m}\cdot\text{s}^{-1} = 25380\,\text{kg}\cdot\text{m}\cdot\text{s}^{-1}.$$

Now consider a system of several bodies. The linear momentum of a system of bodies is defined to be the vector sum of the individual bodies' linear momenta. For simplicity, imagine a collision of just two bodies and suppose there are no uncompensated external forces. Due to Newton's third law the forces they apply on each other are opposite and they last for the same period of time. The first body experiences the force $\boldsymbol{F}_{12}$, the second $\boldsymbol{F}_{21} = -\boldsymbol{F}_{12}$. Changes of their linear momenta will therefore be the same, except the sign

$$\Delta \boldsymbol{p}_1 = \boldsymbol{F}_{12}\Delta t = -\boldsymbol{F}_{21}\Delta t = -\Delta \boldsymbol{p}_2$$

and the change in linear momentum of the system as a whole $\Delta \boldsymbol{p}_1 + \Delta \boldsymbol{p}_2 = \boldsymbol{0}$. The linear momentum of an isolated system of two colliding bodies conserves. We can generalise the result for more than two parts of the system.

!

The linear momentum conservation law
If the effects of external forces on a system are negligible, the total linear momentum of a system is conserved.

$$\Sigma \boldsymbol{F}_{\text{ext.}} = \boldsymbol{0} \Longrightarrow \Sigma \boldsymbol{p} = \text{const.}$$

There is a wide range of applications of the above stated law. It holds even if mechanical energy is not conserved (e.g. inelastic collisions).

Sample problem 5.10 At a fairground, two dodgem cars frontally crash. One is occupied by two stubby adults and its total mass is therefore $m_1 = 400\,\text{kg}$, the second is occupied by a single little boy, so its total mass is $m_2 = 200\,\text{kg}$. Before the crash both the cars have speed of $2\,\text{m}\cdot\text{s}^{-1}$. Assuming there is no loss in mechanical energy in the crash (it is a so called **elastic** collision), determine the speeds of the cars after the crash.
Solution: We have to consider just one component of linear momentum, the one in the direction of motion, as the others are zero before the crash and they as remain zero – the action takes place in a line. Assume the positive direction agrees with the velocity of the first (heavier) car before the crash, thus the velocity of the second car before the crash has a negative sign: $v_1 = v = 2\,\text{m}\cdot\text{s}^{-1}$, $v_2 = -v$, $m_2 = m = 200\,\text{kg}$, $m_1 = 2m$. As we suppose the collision is elastic, both the kinetic energy (there are no changes in the potential energies) and linear momentum of the system are conserved.

$$\begin{aligned} \boldsymbol{p}_1 + \boldsymbol{p}_2 = \boldsymbol{p}_1' + \boldsymbol{p}_2' &\implies 2mv - mv = 2mv_1' + mv_2', \\ E_1 + E_2 = E_1' + E_2' &\implies \frac{1}{2}2mv^2 + \frac{1}{2}mv^2 = \frac{1}{2}2mv_1'^2 + \frac{1}{2}mv_2'^2. \end{aligned}$$

We get

$$\begin{aligned} v &= 2v_1' + v_2', \\ 3v^2 &= 2v_1'^2 + v_2'^2. \end{aligned}$$

From the first equation we express the term $v_2' = v - 2v_1'$ and substitute it into the second. We obtain the quadratic equation (after some manipulation)

$$3v_1'^2 - 2vv_1' - v^2 = 0$$

with two roots, and we find the related v_2' values. There are two possible solutions to our problem, the first:

$$v_1' = v,\ v_2' = -v$$

conforms to the situation when there has been no collision yet, and the second to the situation when the cars have actually collided

$$v_1' = -\frac{v}{3},\ v_2' = \frac{5v}{3}.$$

Remark: Note the velocity vector of the heavier car changed by $v'_1 - v_1 = -v/3 - v = -4v/3$, whilst that of the lighter car by $v'_2 - v_2 = 5v/3 + v = 8v/3$. As the duration of the crash was the same for both cars, the average acceleration of the lighter car must have been twice that of the heavier car. It is similar for regular cars. Although the collisions between regular cars can hardly be considered elastic, the velocity of a lighter car in a crash changes considerably more than the velocity of a heavier one. It follows directly from the law of conservation of linear momentum of a system (which holds even in the case of inelastic collisions): $\Delta \boldsymbol{p}_1 = -\Delta \boldsymbol{p}_2$. Linear momentum is a product of mass and velocity, its change has the same magnitude for each of the crashing cars, so the ratio of the changes of their velocities is inversely proportional to the ratio of their masses. Thus the occupants of the lighter car will be more affected, notwithstanding different safety standards of light and heavy cars.

Any macroscopic collision between real bodies will convert some of their kinetic energy into other forms of energy. Such collisions are called (partially) inelastic. The special case is a **perfectly inelastic collision**, where the bodies stick together and move with the same velocity after the impact, then the loss of the total mechanical energy is maximum. One part of problem 5.12 includes a perfectly inelastic collision.

There are many more conservation laws important in physical processes. At the end of this chapter we will mention a quantity which is the angular counterpart of the linear momentum and it is substantial in the dynamics of rotating rigid bodies. It will be dealt with in more detail later in Physics I. Here, we will discuss it only for point-like bodies and show the first simple application.

! **The angular momentum** of a point-like body with respect to some point is a vector product of the displacement from this point to the body and its linear momentum:

$$\boldsymbol{L} = \boldsymbol{r} \times \boldsymbol{p}.$$

Angular momentum is a vector quantity, its unit is $\mathrm{kg \cdot m^2 \cdot s^{-1}}$ or $\mathrm{J \cdot s}$.
Note that like torque, angular momentum is always defined with respect to some point.

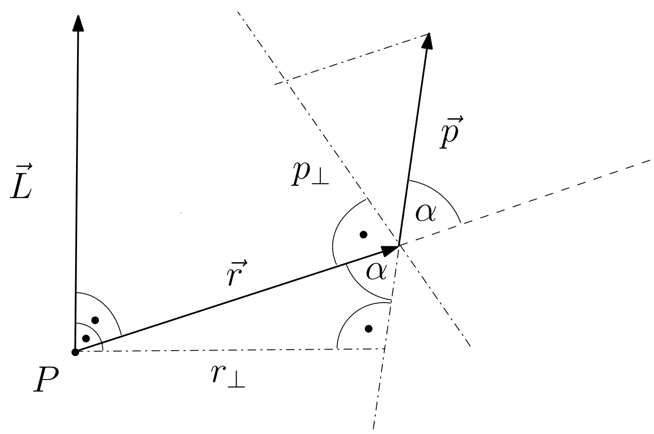


Figure 5.9: *The angular momentum of a particle is a vector product of the displacement from the reference point and its linear momentum*

It follows from the properties of the vector product that the angular momentum is a vector perpendicular to the plane of multiplied displacement and linear momentum and its magnitude is $L = rp\sin\alpha$, where α is the angle between them. Its magnitude can also be expressed as

$$L = pr_\perp,$$

where $r_\perp$ is the component of $\boldsymbol{r}$ perpendicular to $\boldsymbol{p}$, or

$$L = rp_\perp,$$

where $p_\perp$ is the component of $\boldsymbol{p}$ perpendicular to $\boldsymbol{r}$.

There is the angular counterpart to Newton's second law, which expresses the relation between the torque and the rate of change of the angular momentum.

! The time rate of change of the angular momentum of a particle is equal to the net torque acting on the particle:

$$\frac{\mathrm{d}\boldsymbol{L}}{\mathrm{d}t} = \boldsymbol{M}.$$

It is not very difficult to deduce this relation for a moving particle:

$$\begin{aligned}\frac{\mathrm{d}\boldsymbol{L}}{\mathrm{d}t} &= \frac{\mathrm{d}(\boldsymbol{r}\times\boldsymbol{p})}{\mathrm{d}t} = \frac{\mathrm{d}[m(\boldsymbol{r}\times\boldsymbol{v})]}{\mathrm{d}t} = m\left(\boldsymbol{r}\times\frac{\mathrm{d}\boldsymbol{v}}{\mathrm{d}t} + \frac{\mathrm{d}\boldsymbol{r}}{\mathrm{d}t}\times\boldsymbol{v}\right)\\ &= m\left(\boldsymbol{r}\times\boldsymbol{a} + \boldsymbol{v}\times\boldsymbol{v}\right) = (\boldsymbol{r}\times m\boldsymbol{a} + \boldsymbol{0}) = \boldsymbol{r}\times\boldsymbol{F} = \boldsymbol{M}.\end{aligned}$$

Sample problem 5.11 Planets move around the Sun due to gravity. Prove that the angular momentum of an arbitrary planet with respect to the Sun remains conserved along its path. Show that Kepler's second law follows from the conservation of this angular momentum.

Solution: The gravitational force on a planet is a vector in the opposite direction to the displacement from the Sun to the planet (central force motion). Its torque is therefore zero (the torque of force is the vector product of the displacement and the force, $M = rF\sin\varphi$, where φ is the angle between them), so the angular momentum of the planet with respect to the Sun remains conserved.

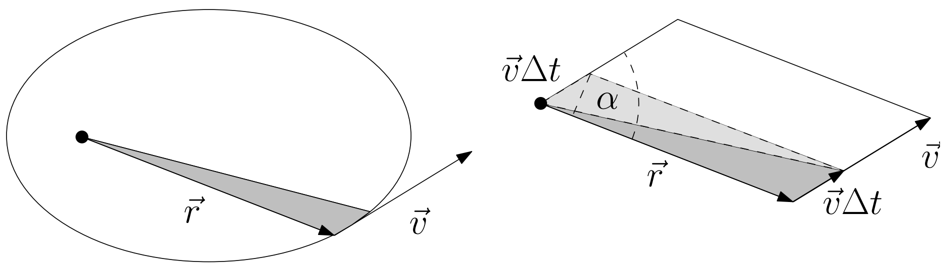


Figure 5.10: *The area swept by the line joining the planet to the Sun in a very short time Δt is at any point equal to the area of the triangle with legs $\boldsymbol{r}$ and $\boldsymbol{v}\Delta t$*

The area swept by the line joining the planet to the Sun in a very short time Δt is at any point equal to the area of the triangle with legs $\boldsymbol{r}$ and $\boldsymbol{v}\Delta t$, and it is half the area of the parallelogram with sides $\boldsymbol{r}$ and $\boldsymbol{v}\Delta t$ (as the time is very short, the curve of path can be replaced with an abscissa). The area of such a parallelogram is base times height, or one side times the other times the sine of the angle between them

$$S_{\mathrm{par.}} = r\cdot v\Delta t\cdot\sin\alpha.$$

It equals the magnitude of the cross product $\boldsymbol{r} \times \boldsymbol{v}\Delta t$. So the area swept by the line joining the planet to the Sun during time Δt is

$$\Delta S = \frac{|\boldsymbol{r} \times \boldsymbol{v}\Delta t|}{2} = \frac{|\boldsymbol{r} \times (m\boldsymbol{v})|\Delta t}{2m} = \frac{|\boldsymbol{r} \times \boldsymbol{p}|\Delta t}{2m} = \frac{|\boldsymbol{L}|\Delta t}{2m}.$$

It follows, that if the angular momentum of the planet with respect to the Sun $\boldsymbol{L}$ remains conserved, a ray from the Sun to a planet sweeps out equal areas in equal times, which we attempted to prove.

If a system does not interact with its environment (or at least if the resultant torque of external forces is zero) the angular momentum of the system remains constant in both magnitude and direction.

Law of conservation of angular momentum
If the net external torque on a system is zero, then the angular momentum of the system remains the same.

$$\Sigma \boldsymbol{M}_{\text{ext.}} = \boldsymbol{0} \Longrightarrow \Sigma \boldsymbol{L} = \text{const.}$$

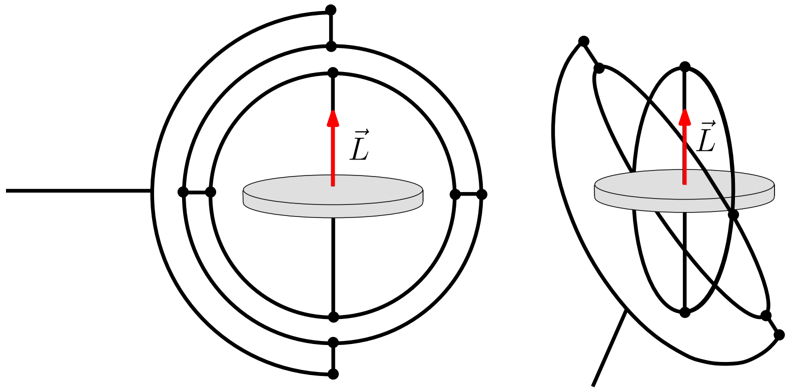


Figure 5.11: *A fast rotating gyroscope keeps its angular momentum* $\boldsymbol{L}$ *and therefore also the direction of its axis of rotation*

A gyroscope, a device designed to maintain a given direction, is based on the conservation of angular momentum. It is a massive wheel which can rotate freely about an arbitrary axis, as it is mounted in a set of gimbals, the friction being minimalised. The rotor is fast-spinning, so it would need a large external torque to change the axis of its rotation. Thus you can change the orientation of the outer frame and the rotor will maintain its spin axis direction. (see figure 5.11, all the joints are swivel joints).

For the same reason, a fast spinning gyroscope in a swivel base (as in figure 5.12) can balance more easily on a stretched spring, as it tends to keep the direction of its axis of rotation. Children's toys, like spinning-tops and yo-yos, work on a similar principle, the gyroscopic effect is also helpful when riding a bike.

We will discuss the physics of rotating bodies in more detail later in Physics I.

Conservation laws of energy, momentum and angular momentum (for any isolated system) are very powerful tools not only in mechanics, but there are generally valid. They are fundamental principles and have far-reaching implications as symmetries of nature which we do not see violated, even at subatomic level. Many problems can be

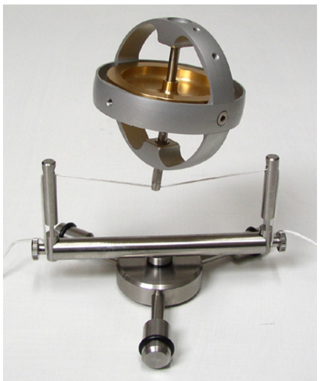


Figure 5.12: *Why a fast-spinning gyroscope seems to balance much more easily on a stretched spring than a non-rotating body?*

solved using conservation laws, as we need only compare the initial and final states of a system, without considering the intermediate states. Later on you will meet some more conservation laws (like conservation of charge, etc.).

The following problems illustrate some classic examples where conservation laws apply.

Problem 5.11 To determine the speed of a projectile we can use a ballistic pendulum. It is a block suspended on a couple of ropes (to prevent its rotation, see the figure 5.13). We fire a projectile against the block and the projectile is captured in it. What was the speed of a projectile weighing $m = 0.536$ g, shot from an air gun, if the block weighing $M = 50$ g came up to $h = 0.2$ m?

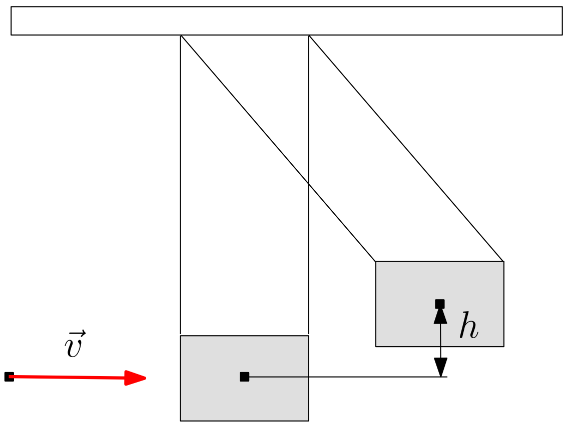


Figure 5.13: *A ballistic pendulum with a projectile fired into it swings to the height h*

Problem 5.12 In July 1975, Apollo space ship attempted to dock with Soyuz space station. The mass of Apollo is $m_A = 18$ tonnes, the mass of Soyuz $m_S = 6.6$ tonnes. Suppose the reference frame connected with Soyuz before the docking procedure. Apollo is approaching Soyuz with the speed $v_A = 0.2\ \mathrm{m \cdot s^{-1}}$. What will be its speed after a head-on collision

a) if the docking manoeuvre is successful (perfectly inelastic collision),
b) if the two spacecrafts collide perfectly elastically?

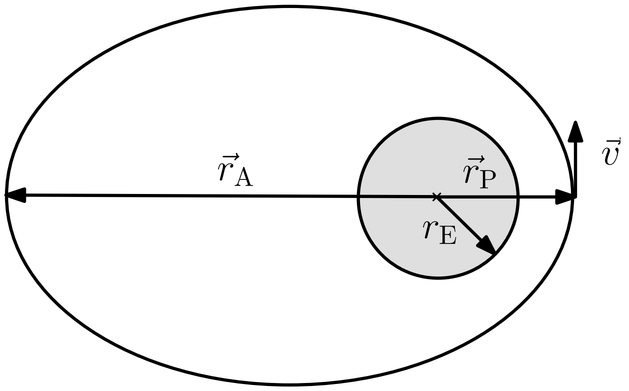


Figure 5.14: *A space ship becomes stabilised along an elliptical orbit before landing*

Problem 5.13 The space ship Apollo becomes stabilised along an elliptical orbit around the Earth before its landing. Its speed at perigee (minimum distance from the Earth) must not be too high, otherwise it will burn up in the atmosphere. At apogee (maximum distance) it has the speed $v_A = 4373 \text{ m} \cdot \text{s}^{-1}$, the distance from the Earth centre $r_A =$ 16 000 km. The radius of the Earth is $r_E = 6370$ km. What will be Apollo's speed at perigee and its height above the Earth's surface there? Make use of the conservation laws of angular momentum and mechanical energy.

Chapter 6

Fluids

Why it is impossible to dive to an arbitrary depth?

Can we lift up a person with a single finger?

Is a glass of water enough to float a boat?

Why does fuel consumption increase rapidly with the speed of a car?

How do airships fly? And how can heavier-than-air aeroplanes fly?

After finishing this chapter you should be able to

- define pressure and know its unit,
- explain Pascal's principle and the operating principle of the hydraulic lever,
- describe pressure distribution in a fluid at rest in gravitational field,
- explain where atmospheric pressure comes from, give examples of the effects it creates, know the approximate value of the standard atmospheric pressure,
- solve simple problems concerning hydrostatic pressure,
- explain where a buoyant force comes from and express its magnitude for a body submerged or partially submerged in a fluid,
- solve simple problems where a buoyant force applies,
- formulate the equation of continuity for the flow of an ideal fluid and explain why it must hold,
- formulate the law of energy conservation for the flow of an ideal fluid (Bernoulli's equation),
- solve simple problems concerning steady irrotational flow of an ideal fluid,
- determine the terminal speed for a body falling with drag.

6.1 Fluid at Rest

In the previous chapters, we attempted to study motion of solid bodies and we assumed they do not change their shapes or volumes when exposed to forces (perhaps, except, in collisions). In contrast, we shall now study substances that can flow, they have no shape of their own and they conform to the boundaries of any container in which we put them. They are called **fluids**. Fluids include **liquids** and **gases**. The role fluids play in our daily lives is as important as that of solid bodies – just think of water, air or blood.

Unlike the molecules of solid bodies, the molecules forming fluids can move easily relative to each other. So a fluid cannot sustain forces tangential to its surface – subjected to such forces it changes its shape. From now on, we shall study fluids not from the point of view of their internal structure, but we shall rather pay attention to their macroscopic characteristics.

Viscosity describes a fluid's internal resistance to flow. Real fluids (except superfluids that can exist only at very low temperatures) have various non-zero viscosities, e.g. water flows easily as it has a lower viscosity, vegetable oil flows less readily as it has a higher viscosity. An **ideal fluid** has no viscosity.

!

A liquid body easily changes its shape, but it keeps its volume. An **ideal liquid** is one that is incompressible and has no viscosity.
A gaseous body easily changes its shape as well as its volume, it conforms to the container into which it is placed. An **ideal gas** is one that is absolutely compressible and has no viscosity.

If you've ever tried diving, you will remember the discomfort or even pain in your ears when you submerge down to two or three metres, which prevents you from diving much deeper without previous training of ear-clearing techniques. This feeling is not eased off if you try to turn round at the same depth, but you can find relief if you go up to the lesser depth or if you apply the right technique to equalise the pressure in your ears.

The force the ideal fluid exerts on any surface submerged in it (e.g. the ear-drums of a free diver) is always normal to the surface (it has no tangential components). This force is called the **compressive force**. It can be measured using a sensor as in figure 6.1. It consists of a box with a close-fitted piston, which can slide inside the box easily. The surface area of a piston is ΔS, it bears on a calibrated spring, the inner part of the box is evacuated. The greater the compressive force, the greater the compression of the spring (we have seen in chapter 5 that the compression of a spring is proportional to the acting force and inversely proportional to the spring constant).

If we turn our sensor around at the same depth, we observe no change in the compression of our spring – the magnitude of compressive force does not depend on the orientation of the piston. It can change, if we shift the sensor to another place in the liquid. If, at the same point of the liquid, we use sensors that have different surface areas, we can see that the compressive force changes, but the ratio of the compressive force to the surface area of the piston holds. It is therefore useful to define a new quantity, called pressure.

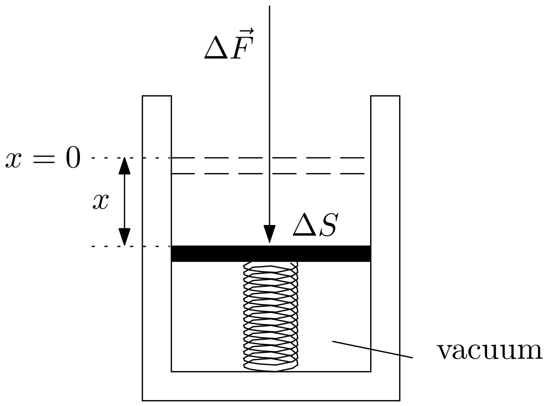


Figure 6.1: *An idealised pressure sensor*

Pressure p is the ratio of the normal force ΔF and the surface ΔS this force is applied on:

$$p = \frac{\Delta F}{\Delta S}$$

Pressure is a scalar quantity, its unit is the pascal (Pa), $\mathrm{Pa} = \mathrm{N} \cdot \mathrm{m}^{-2}$.

Its multiple unit kPa is often used, e.g. the normal atmospheric pressure is 101.325 kPa. We can calibrate our sensors to show pressure ($p = \Delta F/\Delta S = k\Delta x/\Delta S$). From now on, we shall refer to them as pressure sensors.

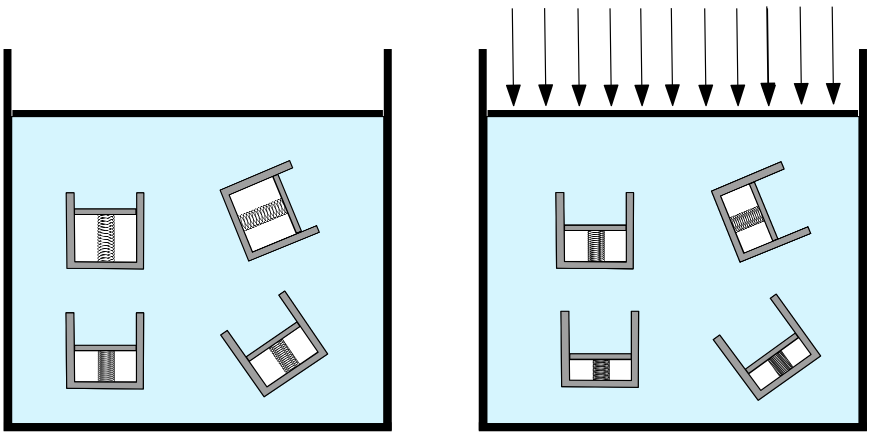


Figure 6.2: *A compressive force may depend on a position in the fluid, but does not depend on an orientation of a pressure sensor. If we compress the piston of the container with fluid, in which several pressure sensors have been placed, we see the pressure at any point grows by the same amount*

Suppose we have a container filled with a fluid and sealed with a movable piston. There are several pressure sensors fixed in the fluid (see figure 6.2). We note the readings of the sensors and then push on the piston. If we compare the new readings with the preceding, we find that the pressure has grown everywhere by the same amount. This experience can be formulated as follows.

Pascal's principle:
Pressure applied to an enclosed fluid is transmitted undiminished to every part of the fluid, as well as the walls of the container.

This principle is the basis of hydraulic and pneumatic arrangements, such as the dental chair, hydraulic brakes in a car (see figure 6.3), presses, pneumatic gears and the like.

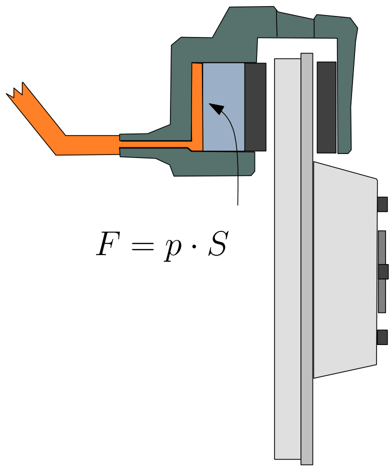


Figure 6.3: *Hydraulic disc brakes in a car. If we press a braking pedal with a relatively small piston, the pressure is transmitted to the larger braking piston thus multiplying the braking force*

Let's study the following simplified example (for the time being, we shall not consider hydrostatic pressure which we will pay attention to later).

Hydraulic lever
In figure 6.4 you can see a simplified model of a hydraulic lever. It is composed of two cylinders with different diameters, connected with a tube. The cylinders are filled with liquid and sealed with two easily sliding pistons. If none of the pistons is loaded, the system is in equilibrium. Now, we put a load on one of the pistons. What load must we place on the other to keep the system in equilibrium?

We can employ Pascal's principle: pressure applied to an enclosed fluid will be transmitted undiminished to every part of the fluid. So (neglecting the hydrostatic pressure) the pressure on both the pistons must be the same $p_1 = p_2$. As $p = F/S$, the ratio of the compressive forces on the second and the first pistons, respectively, must be

$$\frac{F_2}{F_1} = \frac{S_2}{S_1}.$$

If the pistons should be in equilibrium – if they should be at rest, the compressive forces must be compensated by the weights of the loads, so they must have the same ratio.

A similar principle is employed in pneumatic arrangements. As, unlike liquids, air is easily compressible, this compression must be taken into account when designing such devices.

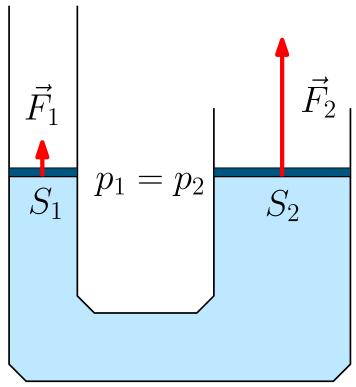


Figure 6.4: *A hydraulic lever principle of operation*

Hydrostatic pressure
Let's have an open tank of water (which we regard as an ideal liquid) at rest. We choose an arbitrary water sample, e.g. a cuboid of horizontal base (see figure 6.5). We mark its base area S and its height h, its volume is then $V = Sh$.

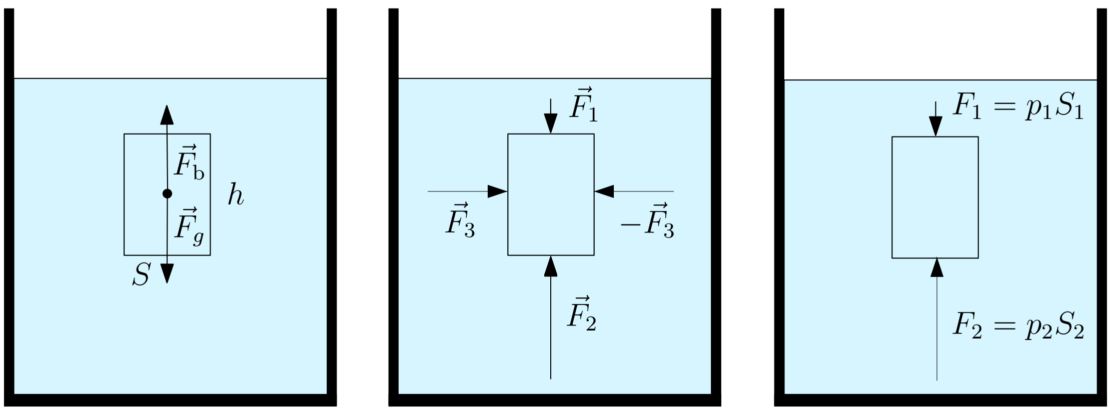


Figure 6.5: *Searching for an expression for hydrostatic pressure as a function of depth*

We shall now discuss the forces acting on our selected sample. As the water in our cuboid is attracted by the Earth, there is the gravitational force pointing downwards, its magnitude is $F_g = m \cdot g = V \cdot \varrho \cdot g = S \cdot h \cdot \varrho \cdot g$, where m is the mass of the water in the cuboid, g is the free-fall acceleration and ϱ is the density of water. Then there is the total force the surrounding water exerts on our sample. As there are no other forces and the cuboid remains at rest, it must be of the same magnitude but opposite to the gravitational force. We call the net upward force of the surrounding fluid a **buoyant force** and denote it as F_b.

We know already that the compressive forces of the fluid are always normal to the surface they act on. It is apparent that the forces on the opposite vertical faces compensate for each other and the compressive force on the lower base must be greater than the compressive force on the upper base by F_b. These forces are $F_2 = p_2 S$ and $F_1 = p_1 S$, respectively. It follows

$$F_2 - F_1 = (p_2 - p_1)S = S \cdot h \cdot \varrho \cdot g,$$

thus the pressure must increase with depth as

$$p_2 = p_1 + h \cdot \varrho \cdot g.$$

On the open surface of the water the total (or absolute) pressure equals the atmospheric pressure. At the depth h, the absolute pressure is greater by the value of the hydrostatic pressure $p_\mathrm{h} = h \cdot \varrho \cdot g$.

! The **hydrostatic pressure** at the depth h in a static liquid with the density ϱ is

$$p_\mathrm{h} = h \cdot \varrho \cdot g,$$

where g is the free-fall acceleration.

The hydrostatic pressure at the depth h in a liquid is independent of the shape of the container (see figure 6.6).

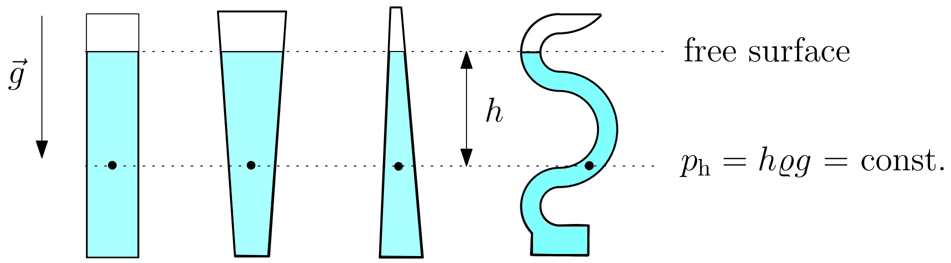


Figure 6.6: *Pascal's vases demonstrate that hydrostatic pressure in a liquid does not depend on the shape of the container in which it is placed. It only depends on the density of the liquid and the depth under the surface*

Buoyant force

Suppose we replace our virtual water cuboid we considered earlier with a cuboid which exactly fills the hole but is made of a different substance. As we have not changed the shape nor the position, the compressive forces on the cuboid's faces must be the same as before. The surrounding liquid will uphold the new body with the same net force as it upheld the water cuboid before. This property of liquids was recognised as early as in the 3 century B.C. by well-known to us already Archimedes.

! **Archimedes' principle:** The buoyant force on a body submerged in a fluid is equal to the weight of the fluid displaced by the body.

$$F_\mathrm{b} = V \cdot \varrho \cdot g,$$

where V is the submerged volume of the body and ϱ the density of the fluid.

Note that the buoyant force applies even if a body is only partially submerged in a fluid, e.g. boats are only partially submerged in water. The buoyant force is then equal to the weight of the displaced fluid, i.e. the weight of the fluid with the same volume as is the volume of the submerged part of the body. Sometimes, even less fluid than the "displaced volume" is needed to demonstrate Archimedes' principle. Consider two slightly conical glasses, one fitting neatly into the other (see figure 6.7). The "displaced volume" V may

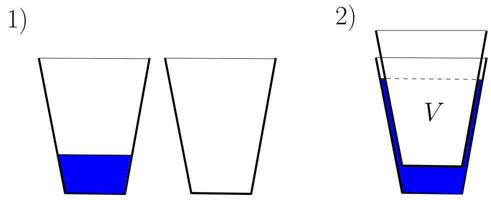


Figure 6.7: *The buoyant force on a body submerged (even partially) in a fluid is equal to the weight of the fluid displaced by the body*

be considerably greater than the real volume of water at disposal. Theoretically, a boat could be held up by a single glass of water, if we had an appropriately shaped container (we must only be aware that fluids are not really continuous at very small scales).

Sample problem 6.1 The world's largest dam is the Three Gorges Dam on the Yangtze river (Long river) in China. It is also the world's largest hydro-electric power station by total capacity, which will be over 18 GW. The concrete dam wall is 2309 metres long and 185 metres high, the water height will be up to $h = 175\,\mathrm{m}$ [23]. Determine the total horizontal thrust (horizontal force) on the main part of the dam which is $l = 2092\,\mathrm{m}$ long. Suppose the dam wall has a rectangular shape.

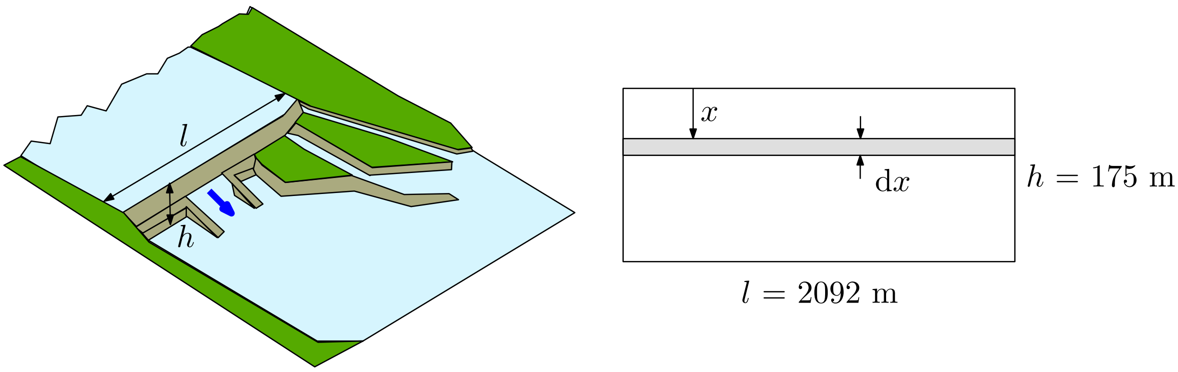


Figure 6.8: *Estimation of the total horizontal force the water of the Three Gorges Dam acts on the main part of the dam*

Solution: Suppose the wall of the dam is vertical. In fact, due to statics requirements, the thickness at base is always greater than the thickness at top of the dam, but it has no effect on the horizontal component of the total force.

As the hydrostatic pressure depends on the depth, we divide the wall to infinitesimally narrow strips of the length l and the height $\mathrm{d}x$, the depth of each strip under the surface we mark as x. Its surface area is therefore $\mathrm{d}S = l\mathrm{d}x$ and the corresponding hydrostatic pressure is $p_\mathrm{h} = x\varrho g$. The compressive force on the strip is therefore $\mathrm{d}F = p_\mathrm{h}\mathrm{d}S = x\varrho gl\mathrm{d}x$. The total force equals the sum of all forces acting on all the strips from the surface to the very bottom of the dam. As there is an infinite number of infinitely narrow strips, the summation turns into integration

$$F = \int \mathrm{d}F = \int_0^h x\varrho gl\mathrm{d}x = \varrho gl \int_0^h x\mathrm{d}x = \varrho gl \left[\frac{x^2}{2}\right]_0^h = \frac{\varrho glh^2}{2}.$$

Numerically we get (supposing the density of the water $\varrho = 1000\,\mathrm{kg\cdot m^{-3}}$)

$$F = \frac{1000 \cdot 9.81 \cdot 2092 \cdot 175^2}{2}\,\mathrm{N} \doteq 3 \cdot 10^{11}\,\mathrm{N}.$$

We could have found the same result even without employing an integration if we had considered the average value of the pressure on the dam. As it grows proportionally to the depth, the average pressure is equal to the half of its maximum value at the bottom. However, if the dam were not a rectangle but e.g. a triangle or a trapezium (as they often fill narrow mountain valleys), the simplified procedure could not be applied.

Torricelli's experiment

Like the hydrostatic pressure in liquids, there is also the aerostatic pressure in gases. Its calculation is more complex, as unlike liquids, gases are compressible. Fortunately, due to the small densities of gases under normal conditions, if we study a gas in a vessel, aerostatic pressure can usually be neglected. However, the pressure of the atmosphere of the Earth cannot be neglected, as an Italian scientist Jan Evangelista Torricelli (1608-1647) demonstrated in his famous experiment in 1644.

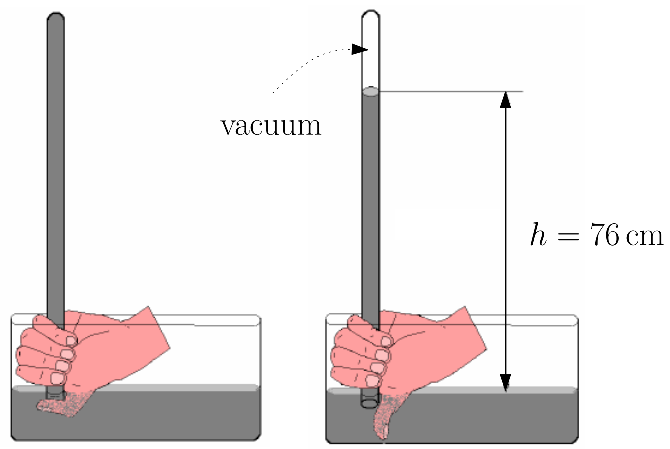


Figure 6.9: *Torricelli's experiment demonstrating atmospheric pressure*

Toriccelli had a basin containing mercury (called quicksilver at thet time) and about 1 m long barometric glass tube open at one end. He filled the tube with mercury and closed the open end with his thumb. He tipped it upside down in a basin of mercury and removed his thumb. He observed that the mercury column only partially descended, stopping at $h = 76\,\mathrm{cm}$ above the level of the mercury in the basin. Torricelli assumed the space in the tube above the column was empty (we call it vacuum), so the pressure there was zero. The pressure of the atmosphere was therefore equal to the hydrostatic pressure at the bottom of the mercury column (at the level of the mercury's open surface in the basin) [24].

Sample problem 6.2 Determine the value of the atmospheric pressure if you know the mercury column in Torricelli's experiment stops at $h = 760$ mm. The density of mercury at 0°C is $\varrho = 13596\ \mathrm{kg\cdot m^{-1}}$, the free-fall acceleration $g = 9.80665\ \mathrm{m\cdot s^{-2}}$.

Estimate the total mass of the atmosphere of the Earth if you know that the mean radius of the Earth is $R_{\mathrm{E}} = 6375\,\mathrm{km}$ and most of the air is concentrated close to its surface (about 90% of the atmosphere by mass is below an altitude of 16 km), so you can consider the gravitational field approximatelly homogenous [25].
Solution: As Torricelli reasoned:

$$p_{\mathrm{a}} = h\varrho g = 0.76 \cdot 13596 \cdot 9.80665\,\mathrm{Pa} = 101.3\,\mathrm{kPa}.$$

The weight of the atmosphere must be equal to the total compressive force of the atmosphere on the Earth (summed magnitudes, not vectors this time). It follows

$$p_{\mathrm{a}} \cdot S = mg \quad \Longrightarrow \quad m = \frac{p_{\mathrm{a}} S}{g} = \frac{p_{\mathrm{a}} \cdot 4\pi R_{\mathrm{E}}^2}{g},$$

where we have applied the formula for the surface area of a sphere. Numerically we get $m \doteq 5 \cdot 10^{18}\,\mathrm{kg}$.

Problem 6.1 A hydraulic lever utilised in a dental chair (the principle of operation as in figure 6.4) has two circular pistons. The larger piston, fixed to the seat, has the diameter $d_1 = 20$ mm, the smaller piston, connected to the pedal, has the diameter $d_2 = 5$ mm. Determine the magnitude of a force needed to apply on the pedal to lift the seat with a patient weighing $m = 160$ kg in total. Disregard the hydrostatic pressure.

Problem 6.2 A drain at the bottom of a swimming pool (5 meters deep) needs cleaning. Could a serviceman use a long hose-pipe, with the open end fixed above the water surface, to breathe? Determine the compressing force acting on his chest area (approximately 40 cm times 50 cm)? The density of water is $\varrho \doteq 1000\ \mathrm{kg \cdot m^{-3}}$.

Problem 6.3 In 2006, an Austrian, Herbert Nitsch, set the world record in freediving (style no-limits) to 183 m. What was the maximum hydrostatic pressure on his body?

What is the pressure at the bottom of the Mariana Trench? The deepest point has been found at 11.034 km below the surface of the sea, the density of water there is approximately $\varrho \doteq 1020\ \mathrm{kg \cdot m^{-3}}$, $g = 9.81\ \mathrm{ms^{-2}}$.

Problem 6.4 Mercury barometers are often calibrated in torrs. The torr is also still used as the unit for blood pressure measurement. The torr is defined as a hydrostatic pressure of 1 mm of the mercury column under normal conditions – the density of mercury is then $\varrho = 13596\ \mathrm{kg \cdot m^{-1}}$, $g = 9.80665\ \mathrm{m \cdot s^{-2}}$. Convert 1 torr to pascals.

Problem 6.5 What is the gauge pressure (the overpressure) of the gas in the vessel connected to the open-tube mercury manometer (see figure 6.10, on the left), if the difference in level between the two columns is $h = 11$ cm?

Problem 6.6 The U-tube in figure 6.10 contains two non-mixing liquids in static equilibrium - water of density $\varrho \doteq 1000\ \mathrm{kg \cdot m^{-3}}$, and oil. The level of the oil is at $h_1 = 10$ cm above the interface, the level of the water at $h_2 = 9.3$ cm above the interface. What is the density of the oil?

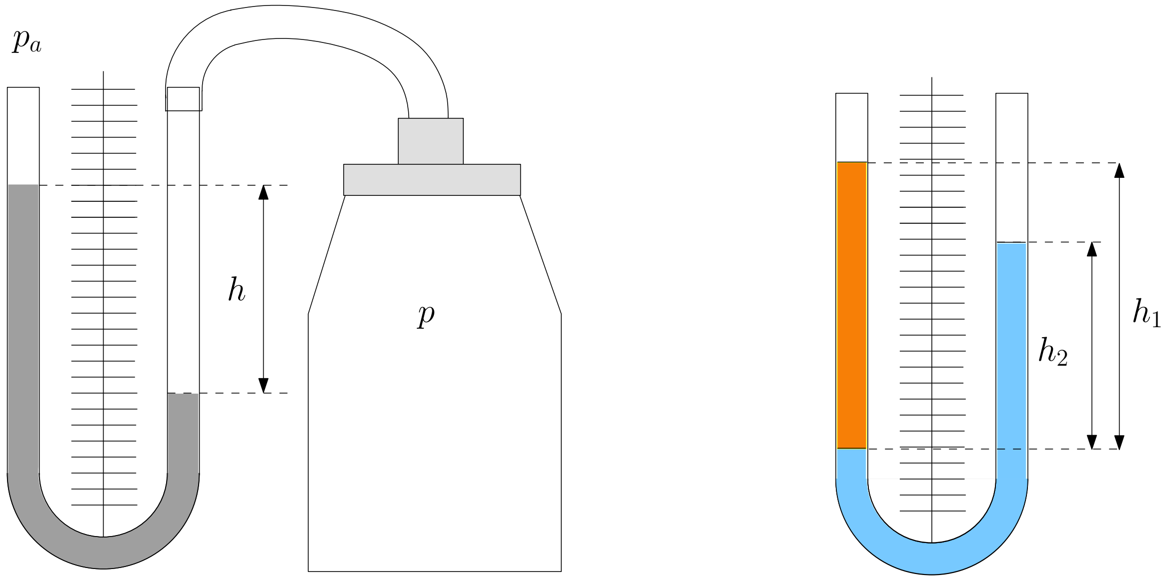


Figure 6.10: *On the left: an open-tube U-manometer used to measure the gauge pressure in the vessel for problem 6.5. On the right: the static equilibrium of two non-mixing liquids in a U-tube, problem 6.6*

Problem 6.7 What is the displaced volume of the the world's largest cruise ship Freedom of the Seas, if its maximum gross tonnage is 158,000 tonnes? The density of sea water is $\varrho \doteq 1020\ \mathrm{kg \cdot m^{-3}}$.

Problem 6.8 The airship - zeppelin LZ 129 Hindenburg, destroyed by a fire while landing in 1937, was filled with hydrogen. The gross lift of the zeppelin was up to 240 tonnes (the useful lift up to 112 tonnes). Suppose the air density approximately $\varrho \doteq 1.29\ \mathrm{kg \cdot m^{-3}}$, the density of hydrogen $\varrho_H \doteq 0.09\ \mathrm{kg \cdot m^{-3}}$ (both at $0\,^\circ\mathrm{C}$). What was the gas volume? Disregard the volume of solid parts of the airship.

Problem 6.9 There is an ice cube floating in a glass of water. What part of it is above the water? How will the level of water in the glass change if the cube melts completely? The density of ice is $\varrho_\mathrm{I} = 900\ \mathrm{kg \cdot m^{-3}}$, the density of water $\varrho = 1000\ \mathrm{kg \cdot m^{-3}}$.

Problem 6.10 There is a wooden trunk floating in water, one third of its volume being above the water level. What is the density of the wood?

Problem 6.11 A rectangular pontoon floats partially submerged in water. Its base area is $S = 20\ \mathrm{m^2}$. When empty, its upper base is at $h_1 = 0.6$ m above the water level. Is it possible to ferry a lorry weighing 8 tonnes, if the water must not exceed the level $h_2 = 0.1$ m below the top of the pontoon?

6.2 Fluid in Steady Motion

Unlike the rigid bodies, the particles of fluids can easily move relative to each other, therefore the motion of fluids can be very complicated. To begin with, we shall discuss the motion of an ideal fluid and make some simplifying assumptions about this motion: we assume incompressible, non-viscous fluid and irrotational steady flow.

To describe the motion of a fluid, we can select tiny elements of it and study their paths. We can add tracers (point-like sources of colour into a liquid or smoke into a gas) to make them visible. We observe the **streamlines** of the fluid. If the flow is steady, the streamlines do not change. New particles are continually being added which follow the same path. As the velocity vector is tangential to the path at every point, the streamlines cannot intersect. A **tube of flow** is defined by the streamlines that form the boundary of the tube, they must go through a closed curve, see figure 6.11.

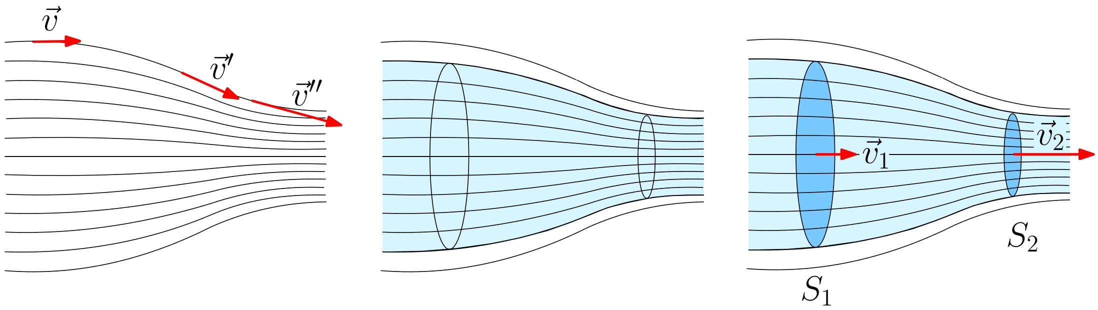


Figure 6.11: *Streamlines, border streamlines of the tube and the stream tube (tube of flow)*

For steady flow, a tube of flow plays the same role as a real pipe – no particles can cross its border to get out of it or inside it from the outside (as each particle follows a streamline and these streamlines cannot intersect with the boundary streamlines of the tube). We shall now define two new quantities suitable to describe flow.

The **mass flow rate** Q_m through a certain cross-section of the tube of flow is the mass of fluid which passes through this cross-section per unit time:

$$Q_m = \frac{\Delta m}{\Delta t}.$$

Its unit is the kilogram per second $\mathrm{kg \cdot s^{-1}}$.

The **volume flow rate** Q_V through a certain cross-section of the tube of flow is the volume of fluid which passes through this cross-section per unit time:

$$Q_V = \frac{\Delta V}{\Delta t}.$$

Its unit is the cubic metre per second $\mathrm{m^3 \cdot s^{-1}}$.

The volume flow rate can be expressed as follows

$$Q_V = \frac{\Delta V}{\Delta t} = \frac{S\Delta l}{\Delta t} = Sv,$$

where Δl is the distance the particles passing via given cross-section cover during the time Δt and v is their speed (the speed of the fluid). The mass flow rate is then equal to

$$Q_m = \frac{\Delta m}{\Delta t} = \frac{\varrho \Delta V}{\Delta t} = \varrho S v,$$

where we assumed $\varrho = \text{const.}$, which holds for an incompressible fluid.

If no particle of fluid can get through the boundary of the tube of flow, all the fluid must subsequently pass through all its cross-sections and the mass flow rate must be constant along the tube. If the fluid is incompressible, the same must hold for the volume flow rate. This is formulated in the equation of continuity.

The equation of continuity

If a fluid flows in a tube of varying cross-section, the mass flow rate is the same everywhere in the tube.

$$S \cdot v \cdot \varrho = \text{const.}$$

Moreover, for an ideal (incompressible) fluid $\varrho = \text{const.}$, so

$$S \cdot v = \text{const.}$$

Bernoulli's equation

Now we try to formulate the law of conservation of energy in a flowing fluid. For simplicity, we suppose the steady flow of an ideal liquid, as in figure 6.12. The derivation for compressible media (like gases) is similar but more complex, as the density can vary along a stream tube. However, if the flow of the gas is sufficiently slow, the variation in density of the gas along each streamline may be ignored and the law we shall derive here can be applied.

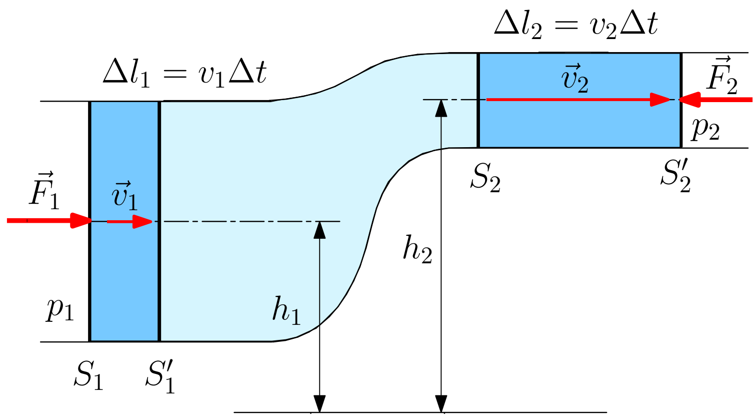


Figure 6.12: *Derivation of Bernoulli's equation*

Suppose the fluid flows at a steady rate from left to right. We shall pay attention to the fluid which at the beginning occupies the volume between the cross-sections S_1 (the pressure here is p_1) and S_2 (the pressure here is p_2). The equation of continuity implies that $S_1 v_1 = S_2 v_2$.

During the short time Δt the fluid moves, the selected volume now occupies the space between S_1' and S_2', where the distance between S_1' and S_1 is $\Delta l_1 = v_1 \Delta t$ and the distance between S_2' and S_2 is $\Delta l_2 = v_2 \Delta t$. Now we shall compare the total energies of our selected volume of the fluid.

As the fluid has left the volume between S_1 and S_1' and it has newly occupied the volume between S_2 and S_2', we compare these two volumes only (the part between the cross-sections S_1' and S_2 was involved at the beginning as well as in the end and the energy here does not change). The mass of the fluid in these two volumes is the same:

$$\Delta m = \varrho S_1 v_1 \Delta t = \varrho S_2 v_2 \Delta t = \varrho \Delta V.$$

The mechanical energy of the fluid in the first is

$$\Delta E_1 = \frac{1}{2}\Delta m v_1^2 + \Delta m g h_1,$$

and in the second

$$\Delta E_2 = \frac{1}{2}\Delta m v_2^2 + \Delta m g h_2.$$

The change in the energy of our system must equal the work done on the system by the rest of fluid. The work done by the fluid on the left is positive, as the compressive force acts in the direction of motion, on the right it is negative as the force acts against the motion.

$$\Delta E_2 - \Delta E_1 = F_1 \Delta l_1 - F_2 \Delta l_2 = p_1 S_1 v_1 \Delta t - p_2 S_2 v_2 \Delta t = (p_1 - p_2)\Delta V.$$

We substitute for the ΔE_2 and ΔE_2 and after some treatment we get

$$\frac{1}{2}\varrho v_1^2 + \varrho g h_1 + p_1 = \frac{1}{2}\varrho v_2^2 + \varrho g h_2 + p_2.$$

Bernoulli's equation is the **equation of energy conservation** for a fluid in motion. The term $\frac{1}{2}\varrho v^2$ represents the density of the kinetic energy of the fluid at the given point and it is also called **dynamic pressure**. The part $\varrho g h$ represents the density of the potential energy of gravity and the pressure p is the density of the pressure potential energy.

Bernoulli's equation

The sum of the densities of the kinetic energy, potential energy of gravity and pressure potential energy remains conserved along a stream tube.

$$\frac{1}{2}\varrho v^2 + \varrho g h + p \ = \ \text{const.}$$

An apparatus fixed at the bottom of an airplane in figure 6.13 is a Pitot-static tube. It is a pressure measuring device used to measure the speed of airplanes or race cars. It consists of a tube pointing directly into the fluid flow and another, opened in the direction normal to the motion (you may notice small holes at the side of the tube). The pressure difference between them is equal to the dynamic pressure of the flowing fluid, so, if the flow is slow enough to disregard compressibility, we get

$$p_2 - p_1 = \frac{1}{2}\varrho v^2.$$

If the speed is too high (particularly supersonic flights), the calibration is more complex.

Many problems concerning non-viscous flow of fluids can be solved employing the equation of continuity and Bernoulli's equation.

Sample problem 6.3 We have an empty sink, whose volume is 10 litres, and we start to fill it with water from a tap. Suppose a steady flow and no air bubbles in the stream. At

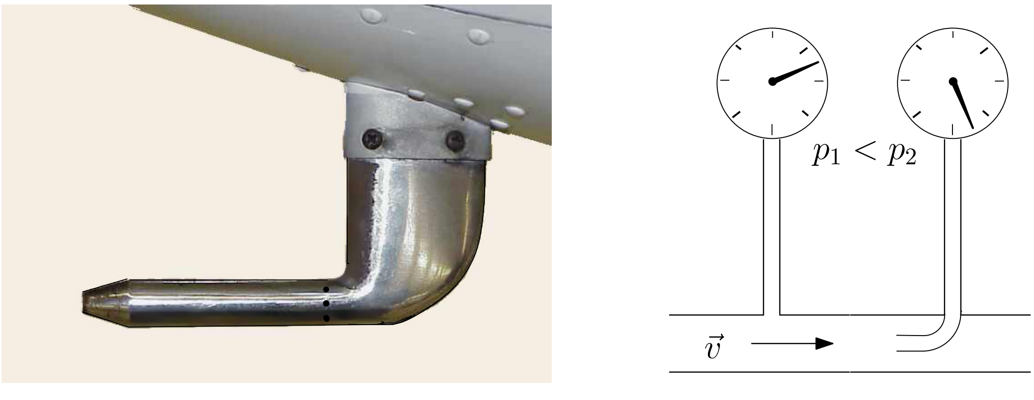


Figure 6.13: *A Pitot tube and the principle of its operation*

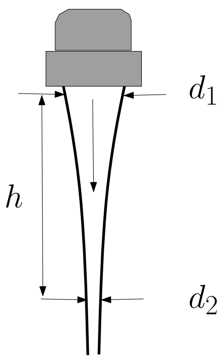


Figure 6.14: *As the water emerging from a tap falls down, it speeds up and the stream narrows*

the top of the stream its horizontal cross-section is $S_1 = 2\ \mathrm{cm}^2$. As the water falls down, it speeds up and the stream narrows, so the cross-section at $h = 5$ cm below the top is only $S_2 = 0.9\ \mathrm{cm}^2$. How long does it take to fill the whole sink?

Solution: We first express the volume flow rate of the water, which must be the same at all horizontal cross-sections: $Q_V = S_1 v_1 = S_2 v_2$. The pressure can be assumed approximately equal to the atmospheric pressure (zero-gauge pressure), so we can write Bernoulli's equation as follows

$$\frac{1}{2}\varrho v_1^2 + \varrho g h_1 = \frac{1}{2}\varrho v_2^2 + \varrho g h_2,$$

so $v_2^2 = 2hg + v_1^2$. We also could have used directly the law of the conservation of energy. Solving the system of the two equations, we obtain

$$Q_V = S_1 S_2 \sqrt{\frac{2hg}{S_1^2 - S_2^2}},$$

so the time necessary to fill up the sink is

$$t = \frac{V}{Q_V} = \frac{V}{S_1 S_2}\sqrt{\frac{S_1^2 - S_2^2}{2hg}}.$$

Numerically

$$t = \frac{0.01}{2 \cdot 10^{-4} \cdot 0.9 \cdot 10^{-4}} \sqrt{\frac{(2 \cdot 10^{-4})^2 - (0.9 \cdot 10^{-4})^2}{2 \cdot 0.05 \cdot 9.81}} \doteq 100\,\mathrm{s}.$$

Sample problem 6.4 The apparatus in figure 6.15 is referred to as Venturi tube. It is designed to measure the flow speed of a liquid from the difference Δh of levels in two manometric tubes. Given the difference Δh and the ratio of the cross-sectional areas S_1/S_2, determine the speed v_1.

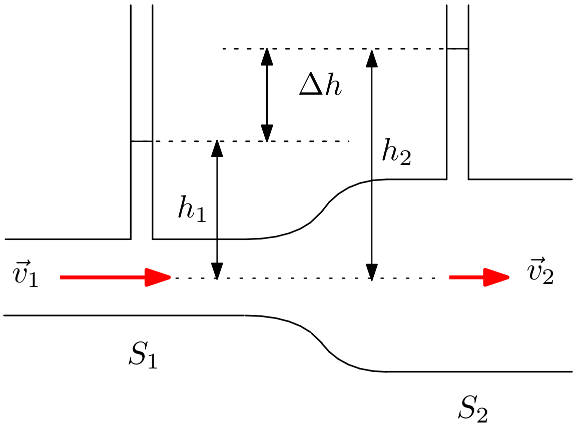


Figure 6.15: *Venturi tube is used to measure the flow speed of a liquid*

Solution: The difference in the levels in the two open manometric tubes indicates the difference in pressure along the tube. The gauge pressure at S_1 must be equal to the hydrostatic pressure $p_1 = h_1 \varrho g$; at S_2 it holds $p_2 = h_2 \varrho g$, so the pressure difference $\Delta p = \Delta h \varrho g$,

$$\Delta h = \frac{\Delta p}{\varrho g}.$$

Bernoulli's equation implies that

$$p_1 + \frac{1}{2}\varrho v_1^2 = p_2 + \frac{1}{2}\varrho v_2^2 \Longrightarrow \Delta p = p_2 - p_1 = \frac{1}{2}\varrho\left(v_1^2 - v_2^2\right).$$

The relation between the speeds v_1 and v_2 can be found using the equation of continuity

$$v_2 = \frac{S_1 \cdot v_1}{S_2}.$$

Finally, we have

$$v_1 = \sqrt{\frac{2g\Delta h}{1 - \left(\frac{S_1}{S_2}\right)^2}}.$$

Real fluid flow

Real fluids have non-zero friction. It is described by a quantity referred to as **viscosity**. Viscosity arises from the shear stress between the sliding layers of a fluid. Most fluids suit the following relation

$$\frac{\mathrm{d}F}{\mathrm{dS}} = \tau = \eta \frac{\mathrm{d}v}{\mathrm{d}y}, \tag{6.1}$$

where τ is the shear stress and η is the dynamic viscosity of a fluid, its unit is $\mathrm{kg \cdot m^{-1} \cdot s^{-1}} = \mathrm{N \cdot s \cdot m^{-2}} = \mathrm{Pa \cdot s}$. Viscosity depends a great deal on temperature. The ratio $\mathrm{d}v/\mathrm{d}y$ is the velocity gradient. The fluids conforming to the formula 6.1 are called Newtonian fluids.

As there is a shear stress between the tube and the boundary layer of a fluid as well, the different layers of the fluid move with different speeds (as you can see in figure 6.16). Any dynamic friction produces the loss of mechanical energy, so, due to viscosity, there is also a drop of pressure along the length of a pipe. However, studying these problems in detail is far beyond the scope of this book.

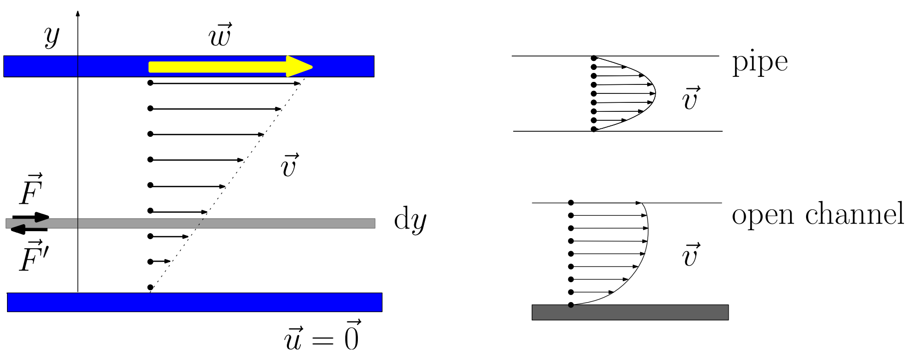


Figure 6.16: *The velocity distribution in a liquid placed between the two parallel planes, the upper one moving to the right (the picture on the left), laminar flow in a circular pipe (top left), and in an open channel (bottom left)*

Some fluids do not agree with the formula 6.1. They are called non-Newtonian fluids and they include pseudoplastic and Bingham fluids. In figure 6.17 you can compare the relation between the shear stress and the velocity gradient for Newtonian and Bingham fluid.

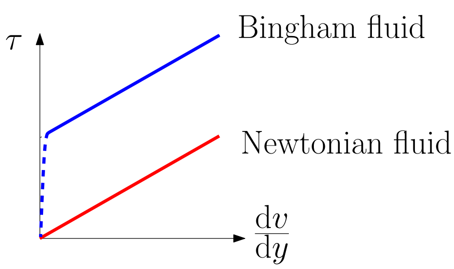


Figure 6.17: *The relation between the shear stress and the velocity gradient for Newtonian and Bingham fluid*

A Bingham fluid is a fluid for which the imposed stress must exceed some critical yield stress to initiate motion. Examples of fluids which behave like Bingham fluids are water suspensions of clay, sewage sludge, some emulsions and greases [26]. If you get into muddy area behaving as a Bingham fluid, you may not be able to get away by yourself.

Problem 6.12 In 1954, Prague was saved from deluge thanks to the Slapy dam, which had just been finished, but yet empty. It could hold up to $0.27\,\mathrm{km}^3$ of water, the maximum volume rate in flood was $2900\,\mathrm{m}^3 \cdot \mathrm{s}^{-1}$ (it was a so called 25-year flood, in the millennial flood in 2002 it was almost twice that much). How long would it take to fill the dam (no water being sent out)? How long would it take at average inflow? The average year-round volume rate is $90\,\mathrm{m}^3 \cdot \mathrm{s}^{-1}$.

Problem 6.13 Water flows steadily with the speed $v_1 = 2\ \mathrm{m} \cdot \mathrm{s}^{-1}$ through a pipe with the cross-section $S_1 = 3\ \mathrm{cm}^2$. At what speed is it emitted from a nozzle with the cross-section $S_2 = 1.5\ \mathrm{cm}^2$? How long does it take to water the green which needs approx. 360 litres of water?

Problem 6.14 In figure 6.18 you can see a device referred to as Prandtl tube. It is designed to measure airspeed and operates on the principle of a Pitot tube. The dynamic pressure is proportional to the difference in the liquid levels in the U-tube. If mercury ($\varrho_{Hg} \doteq 13596\ \mathrm{kg} \cdot \mathrm{m}^{-3}$) is used and the difference is $\Delta h = 8$ mm, what is the speed of the airflow relative to the Prandtl tube? The density of the air is $\varrho \doteq 1.29\ \mathrm{kg} \cdot \mathrm{m}^{-3}$.

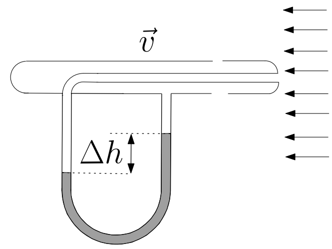


Figure 6.18: *Prandtl tube, problem 6.14*

Problem 6.15 An ideal liquid flows smoothly through a pipe system as in figure 6.15. Show the difference in the liquid levels between the first and the second manometric tubes, if the speed $v_1 = 1.5\ \mathrm{ms}^{-1}$ and the cross-section areas $S_2 = 3S_1$?

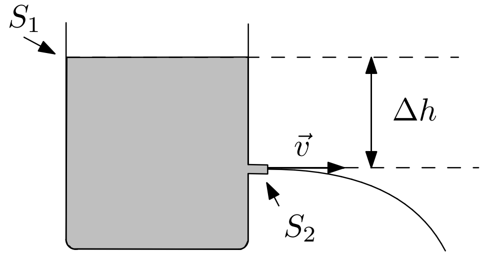


Figure 6.19: *A jet of liquid flowing through a small hole in a tank*

Problem 6.16 Look at figure 6.19. Determine the velocity of the stream emitted through a hole at depth $h = 50$ cm. Assume the open area of the water level in the tank is much larger than the area of the hole and the pressure on both of them is approximately equal to the atmospheric pressure.

6.3 Solid Objects Moving Through a Fluid

If you ride you bike along a horizontal road, you experience a resistive force from the air, opposing your motion. If you go slowly, it is weak and you may not even notice it, but it requires much more effort to maintain a high speed. Likewise, travelling at great speed on a highway, the fuel consumption increases considerably. A sky diver can reduce his landing speed a great deal opening a parachute. All these effect are due to forces that resist the movement of solid objects through a fluid.

An important characteristic here is the Reynolds number. It is the ratio of inertial forces to viscous forces and shows which of them play a dominant role in drag. It can be expressed as follows

$$Re = \frac{v\varrho d}{\eta},$$

where d is the characteristic length of the body, v is its speed relative to the fluid, ϱ is the density of the fluid and η its dynamic viscosity. The Reynolds number also indicates if the flow is laminar or turbulent, e.g. in pipes the critical value is about 2300.

At small Reynold numbers ($Re \ll 1$, small objects moving at low speeds) the drag force arises mainly because of viscous forces in a fluid. The drag force opposing motion is then proportional to the relative speed, and it is referred to as **Stokes's drag**:

$$\boldsymbol{F}_\mathrm{d} = -k\eta l\boldsymbol{v},$$

where k is the proportionality constant which depends on the shape of the body, η is dynamic viscosity of a fluid and l is the characteristic length of the body. As an example, for a sphere it holds $\boldsymbol{F}_\mathrm{d} = -6\pi\eta r\boldsymbol{v}$. A small sphere falling through a highly viscous fluid can experience drag corresponding to this rule, like that of problem 6.17. As the Reynolds number for which this relation holds is very small, its applicability is highly restricted.

At large Reynold numbers (approx. $Re > 1000$, large bodies or higher speeds) the drag is mainly due to the inertia of particles of a fluid and it is proportional to the square of the relative speed. It is known as **Newton's drag** equation:

$$\boldsymbol{F}_\mathrm{d} = -\frac{1}{2}CS\varrho v\boldsymbol{v},$$

where C is the proportionality constant which depends on the shape of the body, S is the frontal area (effective cross-sectional area) of the body and ϱ is the density of the fluid. It applies when animals, cars or most other objects move through the air or water. We have already mentioned this force in chapter 4 when discussing a ballistic curve.

Since drag grows as speed grows, there is always a particular speed which the falling body reaches if it falls long enough and it below moves steadily at this speed. It is called the **terminal speed** and it does not depend on the original speed of the body, no matter if it was higher or lower than the terminal speed. It can be easily determined using Newton's first law.

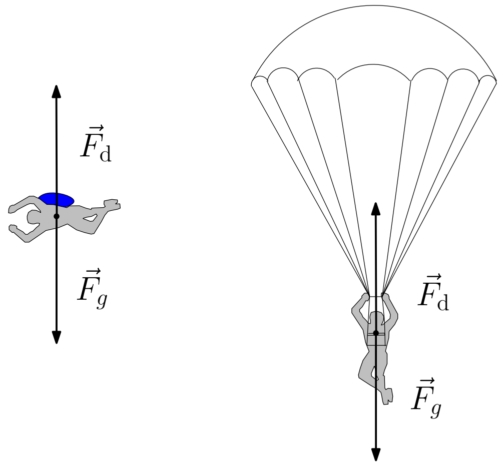


Figure 6.20: *A skydiver falling at terminal speed – the drag force has the same magnitude as gravity*

Sample problem 6.5 Find the terminal speed of a sky diver weighing 100 kg (including a parachute), while his parachute is not yet open. The density of the air is $\varrho = 1.29\ \mathrm{kg \cdot m^{-3}}$, the effective cross-sectional area $S = 0.7\,\mathrm{m}^2$, the proportionality constant $C = 0.5$. What is his landing speed with an open parachute, $S' = 36\,\mathrm{m}^2$, $C' = 1.3$?
Solution: There are two forces acting on the sky diver (disregarding a buoyant force which is quite small). It is the gravity $\boldsymbol{F}_g$ and the drag force $\boldsymbol{F}_\mathrm{d}$ opposite to it. The speed of the sky diver will become stable when these two are in equilibrium, so $\boldsymbol{F}_g + \boldsymbol{F}_\mathrm{d} = 0$. We substitute for the forces and obtain

$$mg = \frac{1}{2}CS\varrho v^2 \qquad \Longrightarrow \qquad v = \sqrt{\frac{2mg}{CS\varrho}}.$$

Numerically we get

$$v = \sqrt{\frac{2 \cdot 100 \cdot 9.81}{0.5 \cdot 0.7 \cdot 1.29}} \doteq 66\ \mathrm{m \cdot s^{-1}} = 237\ \mathrm{km \cdot h^{-1}}$$

if the parachute is not yet open and

$$v = \sqrt{\frac{2 \cdot 100 \cdot 9.81}{1.3 \cdot 36 \cdot 1.29}} \doteq 5.7\ \mathrm{m \cdot s^{-1}} = 20.5\ \mathrm{km \cdot h^{-1}},$$

the terminal speed with the open parachute.

Thanks to the drag heavier-than-air aeroplanes can fly. In figure 6.21 you can see the forces acting on a plane. They include gravity $\boldsymbol{F}_g$, aerodynamic force $\boldsymbol{F}_\mathrm{a}$, which is further split into dynamic lift $\boldsymbol{F}_\mathrm{al}$ and drag $\boldsymbol{F}_\mathrm{ad}$, and the thrust $\boldsymbol{F}_\mathrm{t}$. The thrust is generated by a propulsion system (it may be a jet engine, using a reactive force of hot gas rushing out of the plane, or it may employ some kind of interaction with the surrounding air).

The air flows along a wing and it is diverted diagonally down, thanks to Newton's third law it at the same time pushes the plane diagonally up. The viscosity of the

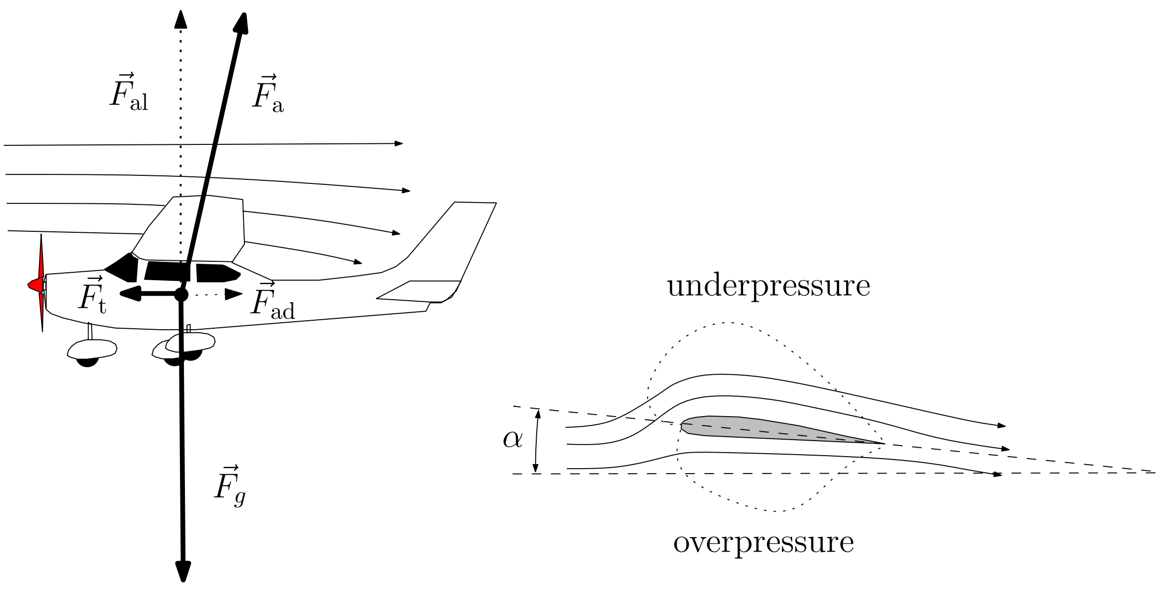


Figure 6.21: *The forces acting on an airplane, the air-flow along a wing and the pressure distribution around it*

air and the existence of a boundary layer plays an important role here. The lift of a wing is proportional to the amount of air diverted down times the vertical velocity of that air. Look at figure 6.21 on the right, we see the air-flow along the wing and the pressure distribution around it. The distance of the dotted curve from the wing indicates the magnitude of gauge pressure at a given point. The pressure distribution depends predominantly on the inclination of a wing. A simple plain board would have a quite similar pressure distribution as you can see from the picture, therefore the rectangular profiles can be used in aeromodelling, or symmetric wings on acrobatic aeroplanes.

The final force the air exerts on a wing is the integration of the pressure over the surface of the wing

$$\boldsymbol{F}_{\mathrm{a}} = -\int p \mathrm{d}\boldsymbol{S}$$

where $\mathrm{d}\boldsymbol{S}$ is an infinitesimal element of the area times the normal unit vector pointing out of the wing.

The key laws applying to airflow around the wing are the continuity equation, the law of energy conservation and the law of conservation of linear momentum. However, the theory of flight is complex and our introduction is very simplified. Some popular but credible discussions of the topic can be found at [27].

Problem 6.17 A falling sphere viscosimeter can be used to measure the viscosity of a liquid. What is the viscosity of an unknown liquid, in which a falling sphere attains the terminal speed $v = 1.8\ \mathrm{mm \cdot s^{-1}}$? The diameter of the sphere is $d = 2\,\mathrm{mm}$, its density $\varrho = 2500\ \mathrm{kg \cdot m^{-3}}$, the density of the liquid is found as $\varrho' = 1260\ \mathrm{kg \cdot m^{-3}}$. Check if the result you get makes it possible to apply Stokes's law.

Problem 6.18 The increased danger of injury and damage by large hailstones is induced not only by their greater weight, but also by their considerably higher terminal speed. Find the terminal speed of a spherical hailstone (suppose the proportionality constant

$C = 0.5$) with the radius $r_1 = 2\,\mathrm{mm}$ and with the radius $r_2 = 2\,\mathrm{cm}$. The density of the ice is $\varrho = 920\ \mathrm{kg \cdot m^{-3}}$, the density of the air $\varrho' = 1.29\ \mathrm{kg \cdot m^{-3}}$, disregard the buoyancy.

Problem 6.19 It is raining in still air. A train goes at the speed $u = 60\ \mathrm{km \cdot h^{-1}}$ and the rain drops strike the windows inclined at an angle of $\alpha = 60°$ from horizontal. Determine the speed and the radius of a raindrop. The density of the water is $\varrho = 1000\ \mathrm{kg \cdot m^{-3}}$, the density of the air $\varrho' = 1.29\ \mathrm{kg \cdot m^{-3}}$.

Answers to Problems

Chapter 1

Sample problem 1.1
1/2 pound = 227 g = 0.227 kg; 1/4 pound = 113.5 g = 0.1135 kg;
4 tablespoons = 60 ml = $6 \cdot 10^{-5}$ m^3; 1 tablespoon = 15 ml = $15 \cdot 10^{-6}$ m^3;
1/2 teaspoon = 2.5 ml = $2.5 \cdot 10^{-6}$ m^3; 1 teaspoon = 5 ml = $5 \cdot 10^{-6}$ m^3;
10 fl. oz. = 300 ml = $3 \cdot 10^{-4}$ m^3; 8" = 20.32 cm = 0.2032 m;
3 a 1/2 h. = 12600 s; 4 h. = 14400 s; 300 °F = 149 °C

Problem 1.1 $5 \cdot 10^{-5}\,\mathrm{m}$

Problem 1.2 $36.1\,\mathrm{m} \cdot \mathrm{s}^{-1}$

Problem 1.3 $0.44

Problem 1.4 $P = Fv = mgs/t$, 1 horsepower = 746 W

Problem 1.5 Pluto, 1.032

Problem 1.6 The area swept by a line joiningthe planet to the Sun during a very short time Δt will be $\Delta S = r \cdot v\Delta t$, where r is the instantaneous distance from the Sun. Earth: 1.04, Mercury 1.52

Problem 1.7 Mercury: 0.244 year, Pluto: 251 years

Problem 1.8 It will come back in 2061, a=17.79 AU, b=4.53 AU

Problem 1.9 Approx. $30\,\mathrm{km} \cdot \mathrm{s}^{-1}$

Problem 1.10 2642 km

Problem 1.11 Approx. $0.9a_{\mathrm{E-M}} = 3.44 \cdot 10^8\,\mathrm{m}$

Problem 1.12 The Moon's gravity is one-sixth that of the Earth's.

Chapter 2

Problem 2.1 $\boldsymbol{A} + \boldsymbol{B} = (0, 60, 30)$.

Problem 2.2 $A = 53.85$

Problem 2.3 $\alpha = 68°12'$, $\beta = 56°9'$ a $\gamma = 42°2'$

Problem 2.4 a) $T = \frac{W}{2}$, b) $T = \frac{W}{3}$

Problem 2.5 a) $F = 1\,\text{kN}$, b) $F = 1.93\,\text{kN}$

Problem 2.6 Friction, $\mu_s \geq 0.1$

Problem 2.7

$$T = \frac{\mu_k mg}{\cos\alpha + \mu_k \sin\alpha}$$

Problem 2.8 $m \in (M\sin\alpha - M\mu_s\cos\alpha, M\sin\alpha + M\mu_s\cos\alpha)$, $m \in (0.4; 5.6)$ kg

Problem 2.9 Andrew will beat John by almost 42 seconds.

Problem 2.10 a) $\boldsymbol{A} \cdot \boldsymbol{B} = 10$, b) $\boldsymbol{A} \times \boldsymbol{B} = (-4.8, -4)$, c) $\varphi = 44°25'$

Problem 2.11 No, we have only 4 equations, but 6 unknown coordinates. We know that $\boldsymbol{A}$ and $\boldsymbol{B}$ lie in xy-plane and the angle between them is 45°.

Problem 2.12 $\boldsymbol{T}_1 = (0, -435, 1050)\,\text{N}$, $\boldsymbol{T}_2 = (0, 435, 1050)\,\text{N}$, $T_1 = T_2 = 1137\,\text{N}$.

Problem 2.13 $d = 2\,\text{m}$, independently of the momentary inclination.

Problem 2.14

$$N_A = W\cos\alpha\left(\frac{1}{2} - \frac{h}{l}\tan\alpha\right) = 5\,\text{kN}, \quad N_B = W\cos\alpha\left(\frac{1}{2} + \frac{h}{l}\tan\alpha\right) = 7.5\,\text{kN}.$$

a)

$$\mu = \frac{\tan\alpha}{\frac{1}{2} - \frac{h}{l}\tan\alpha} = 0.91,$$

b)

$$\mu = \frac{\tan\alpha}{\frac{1}{2} + \frac{h}{l}\tan\alpha} = 0.60$$

c)

$$\mu = \tan\alpha = 0.36.$$

Chapter 3

3.1

(a)	$2x+2$	(g)	$-\dfrac{\cos x}{\sin^2 x}$
(b)	$-\dfrac{1}{x^2}-\dfrac{2}{x^3}$	(h)	$\dfrac{1}{\cos^2 x}$
(c)	$\dfrac{1}{2\sqrt{x}}$	(i)	$\dfrac{x}{\sqrt{x^2+1}}$
(d)	$-\dfrac{1}{2\sqrt{x^3}}$	(j)	$-\dfrac{1}{x}$
(e)	$2\cos 2x$	(k)	$2\cos(2x-3)$
(f)	$-\sin 2x$	(l)	$-\mathrm{e}^{1-x}$

Problem 3.2 Yes, even 20 m ahead; $v(2) = 16\,\mathrm{m}\cdot\mathrm{s}^{-1}$; negative acceleration $a = 2\,\mathrm{m}\cdot\mathrm{s}^{-2}$.

Problem 3.3

$$v_{\max} = 57.7\ \mathrm{m}\cdot\mathrm{s}^{-1}, \qquad a = \left(\frac{39.24}{\mathrm{e}^{0.34t}+2+\mathrm{e}^{-0.34t}}\right)\ \mathrm{m}\cdot\mathrm{s}^{-2}.$$

Problem 3.4 $t = \frac{\pi}{2}$ s, $v_{\max} \doteq 6.45\,\mathrm{m}\cdot\mathrm{s}^{-1}$.

Problem 3.5

$$\frac{\mathrm{d}\mu_2}{\mathrm{d}a} = \frac{-\mu_1 l\,(1+\tan^2\alpha)}{[l-a\,(1-\mu_1\tan\alpha)]^2}.$$

As it is negative for $a \in \langle 0,\, l\rangle$, the function is descending here.

Problem 3.6 Hint: (f) use the identity $\cos 2x = \cos^2 x - \sin^2 x = 2\cos^2 x - 1$; (j), (k) and (l) use the per partes method. Results:

(a)	$\dfrac{x^3}{3}+x^2-3x+C$	(g)	$-\mathrm{e}^{1-x}+C$
(b)	$\ln\lvert x\rvert-\dfrac{1}{x}+C$	(h)	$-\ln\lvert\cos x\rvert+C$
(c)	$\dfrac{2}{3}\sqrt{x^3}+C$	(i)	$\dfrac{1}{3}\sqrt{(x^2+1)^3}+C$
(d)	$\dfrac{1}{2}\ln\lvert 2x-1\rvert+C$	(j)	$\dfrac{x^4}{4}\ln x-\dfrac{x^4}{16}+C$
(e)	$-\dfrac{1}{2}\cos 2x+C$	(k)	$-x\cos x+\sin x+C$
(f)	$\dfrac{x}{2}+\dfrac{1}{4}\sin 2x+C$	(l)	$\dfrac{\mathrm{e}^x}{2}(\sin x+\cos x)+C$

Problem 3.7 Substitution $t = \sqrt{x^2+1}$, yields $\sqrt{2}-1$.

Problem 3.8 Substitution $t = \sin x$, yields $\frac{1}{3}$.

Problem 3.9 Substitution $t = \mathrm{e}^{0.17t} + \mathrm{e}^{-0.17t}$, yields

$$s = 339 \ln \frac{\mathrm{e}^{0.17t} + \mathrm{e}^{-0.17t}}{2} = 339 \ln \cosh(0.17t).$$

Chapter 4

Problem 4.1

$$v = \sqrt{\mu g r} = 9.4\,\mathrm{m} \cdot \mathrm{s}^{-1} = 33.8\,\mathrm{km} \cdot \mathrm{h}^{-1}$$

Problem 4.2

$$f = \frac{1}{2\pi} \sqrt{\frac{g \tan \alpha}{r}} = 0.15\,\mathrm{Hz}$$

Problem 4.3

$$v = \sqrt{g r \tan \alpha + \frac{a_{\mathrm{t}} r}{\cos \alpha}} = 32.3\,\mathrm{m} \cdot \mathrm{s}^{-1} = 116\,\mathrm{km} \cdot \mathrm{h}^{-1},\ \text{an ordinary train } 89\,\mathrm{km} \cdot \mathrm{h}^{-1}.$$

Problem 4.4 $465\,\mathrm{m} \cdot \mathrm{s}^{-1}$, relative to the Sun $29.8\,\mathrm{km} \cdot \mathrm{s}^{-1}$.

Problem 4.5 The altitude is 20 200 km.

Problem 4.6 Above the equator only, the altitude approx 36 000 km.

Problem 4.7 $t = \frac{2v_0}{g}, H = \frac{v_0^2}{2g}$

Problem 4.8 $t = 1.23$ s

Problem 4.9 $\alpha = 45°$

Problem 4.10 $\alpha = 76°$

Problem 4.11 Find the intersection of the parabolic path of the stone and the inclined plane $y = x \tan \beta$. The result is $t = 2.07$ s, $l = 29.7$ m

Problem 4.12

$$x^2 + y^2 + (z - h + 1/2 g t^2)^2 = v^2 t^2.$$

Chapter 5

Problem 5.1 $30\,\mathrm{kJ}$

Problem 5.2 $58\,\mathrm{m}$

Problem 5.3

$$A = \int_0^h g\,(100 - 3s)^2\,\mathrm{d}s \ldots \quad A = 716\,\mathrm{kJ}$$

Problem 5.4

$$E_\mathrm{p} = \int_{R_\mathrm{Z}}^{R_\mathrm{Z}+h} \kappa\,\frac{mM}{r'^2}\,\mathrm{d}r' = \kappa mM \left[-\frac{1}{r'}\right]_{R_\mathrm{Z}}^{R_\mathrm{Z}+h} = \kappa mM \left(\frac{1}{R_\mathrm{Z}} - \frac{1}{R_\mathrm{Z}+h}\right),$$

$E_\mathrm{k} = \frac{1}{2}mv^2$, celkově $E = E_\mathrm{p} + E_\mathrm{k} = 4.27\cdot 10^{10}\,\mathrm{kJ}$

Problem 5.5 $P(3) = 3.75\,\mathrm{W}$, $\overline{P} = 1.88\,\mathrm{W}$

Problem 5.6 The overload equals the centripetal acceleration, $a = 2gh/R$, $a = 1.83g$.

Problem 5.7

$$h = \frac{5gR - v_0^2}{2g}$$

Problem 5.8

$$\alpha = \arcsin\frac{P}{mgv} = 31°19'.$$

Problem 5.9

$$l = h\left(\frac{1}{\mu_\mathrm{D}} - \cot\alpha\right) = 80\,\mathrm{cm}.$$

Problem 5.10 It is risky, the maximum elongation is approx. 13.3 m.

Problem 5.11

$$v = \frac{m+M}{m}\sqrt{2gh} = 187\,\mathrm{m\cdot s^{-1}}$$

Problem 5.12

$$\text{a)}\quad v = \frac{m_\mathrm{A} v_\mathrm{A}}{m_\mathrm{S} + m_\mathrm{A}} = 0.146\,\mathrm{m\cdot s^{-1}}$$

$$\text{b)}\quad v = \frac{2m_\text{A}v_\text{A}}{m_\text{S} + m_\text{A}} = 0.293\,\text{m}\cdot\text{s}^{-1}$$

Problem 5.13 The law of conservation of angular momentum and the law of conservation of mechanical energy imply $v_P = 7028\ \text{m}\cdot\text{s}^{-1}$, the altitude $h = 3581\text{km}$.

Chapter 6

Problem 6.1

$$F_2 = \frac{mgd_2^2}{d_1^2} = 98\ \text{N}.$$

Problem 6.2 $F = p_\text{h}S = h\varrho gS \doteq 9810$ N, he could not breathe.

Problem 6.3 $p_\text{h1} = 1.83$ MPa, $p_\text{h2} = 110$ MPa.

Problem 6.4 1 torr = 133.3 Pa.

Problem 6.5 $p = 14.7$ kPa.

Problem 6.6 $\varrho_1 = \varrho_2 h_2/h_1 = 930\ \text{kg}\cdot\text{m}^{-3}$.

Problem 6.7 $V = m/\varrho = 1.55\cdot 10^5\ \text{m}^3$

Problem 6.8 $V = m/(\varrho - \varrho_\text{H}) = 2\cdot 10^5\ \text{m}^3$.

Problem 6.9

$$\Delta V = V_\text{I} - V = V_\text{I} - \frac{9}{10}V_\text{I} = \frac{1}{10}V_\text{I}.$$

The level will remain the same.

Problem 6.10 $F_\text{b} = G \Longrightarrow \varrho_D = (2/3)\varrho_\text{V} = 667\ \text{kg}\cdot\text{m}^{-3}$.

Problem 6.11 The maximum possible load is $G = \Delta F_\text{b} = \varrho\Delta hS\ g \Longrightarrow m = \varrho\Delta hS =$ 10 tonnes, so it is possible.

Problem 6.12 Approx. 26 hours, almost 35 days.

Problem 6.13 $v = 4\ \text{m}\cdot\text{s}^{-1}$, $t = 10$ min.

Problem 6.14 $v = \sqrt{2gh\frac{\varrho_{Hg}}{\varrho}} \doteq 41\ \text{m}\cdot\text{s}^{-1}$.

Problem 6.15

$$\Delta h = \frac{v_1^2}{2g}\left(1 - \frac{S_1^2}{S_2^2}\right) = 0.1\ \text{m}.$$

Problem 6.16 $v = \sqrt{2gh} = 3.1 \text{ m} \cdot \text{s}^{-1}$.

Problem 6.17

$$\eta = \frac{d^2 (\varrho - \varrho') g}{18v} = 1.5 \text{ Pa} \cdot \text{s}, \quad Re = 3 \cdot 10^{-3}.$$

Problem 6.18

$$v = \sqrt{\frac{8\varrho g r}{3C\varrho'}}, \quad v_1 = 8.6 \text{ m} \cdot \text{s}^{-1}, \quad v_2 = 27.3 \text{ m} \cdot \text{s}^{-1}.$$

Problem 6.19

$$v = u \cot \alpha = 9.6 \text{ m} \cdot \text{s}^{-1}, \quad r = \frac{3v^2 C \varrho'}{8g\varrho} = 2.3\,\text{mm}.$$

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